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The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

unit 27 multiple integrals

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Unit 27
Multiple integrals

Prepared for the Course Team
by John Berry

The Open University

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Introduction

This unit is about the integration of functions of two variables over areas in a plane and the integration of functions of three variables over volumes in three-dimensional space. These types of integrals are called surface integrals and volume integrals respectively. They are two examples of integrals called multiple integrals. A related type of integral is the line integral, which was introduced in *Unit 26*.

Surface and volume integrals are extremely useful mathematical tools for applied mathematicians. This unit gives examples of the use of multiple integrals in mechanics, although the actual physical interpretation of the properties defined here will be postponed until a later unit. The aim of this unit is to give you plenty of practice at evaluating multiple integrals so that for later mechanics units you will have the results of the multiple integrals which are needed in the mechanics applications.

This unit contains four sections. Section 1 begins by defining a surface integral and then we see how to evaluate surface integrals over regions of the x,y-plane.

Section 2 contains the television programme for this unit. The television programme shows that the multiple integrals can be used in mechanics to evaluate the mass and moment of inertia of a body. Experiments show the physical significance of these quantities. However, the mechanics associated with the motion of bodies, as opposed to particles, will be studied in a later unit. In the television programme we also show that a change of co-ordinates sometimes makes the evaluation of multiple integrals more straightforward. The text picks up this idea and shows how to transform and evaluate surface integrals using polar co-ordinates. The audio-tape activity is in this section.

Section 3 defines volume integrals and shows how to evaluate volume integrals using Cartesian co-ordinates.

Section 4 introduces two new three-dimensional co-ordinate systems: cylindrical polar co-ordinates and spherical polar co-ordinates. We then use these co-ordinates to evaluate volume integrals over regions with axial and spherical symmetry.

Study guide

The sections should be studied in the order in which they appear. Before watching the television programme you should have studied up to page 18. Sections 1, 2.1 and 3 are crucial for a proper understanding of later mechanics units. After studying this unit you should have had plenty of practice at evaluating surface and volume integrals and you will be ready to use them in the units that follow.

1 The surface integral

1.1 What is a surface integral?

You should already be familiar with the idea of integrating a continuous function over an interval $[a, b]$ of the real line, i.e. you can evaluate integrals of the form $\int_a^b f(x)\,dx$ which are called *definite integrals.* The definite integral is defined as the limit of a sum in the following way:

We divide the interval $[a, b]$ into N subintervals and in each subinterval we select a point:

x_1 in the first subinterval,

x_2 in the second subinterval,

⋮

x_i in the ith subinterval, and so on (see Figure 1).

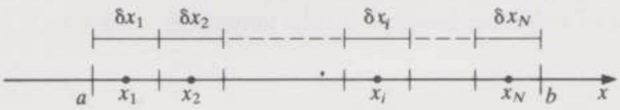


Figure 1

Then we form the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \cdots + f(x_i)\delta x_i + \cdots + f(x_N)\delta x_N$$

where δx_i is the width of the ith subinterval. This is the sum of the areas of the rectangles as in Figure 2.

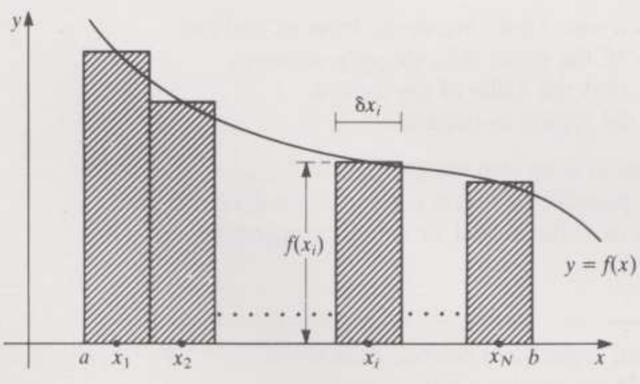


Figure 2

The limit of this sum as we increase the value of N indefinitely and the lengths of all the subintervals go to zero is defined to be the **definite integral of f from a to b** and is written

$$\int_a^b f(x)dx = \lim \sum_{i=1}^{N} f(x_i)\delta x_i.$$

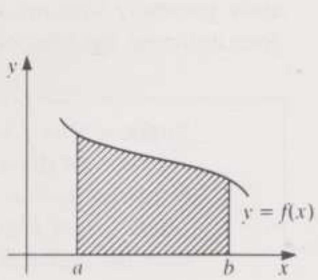


Figure 3

Geometrically, if $f(x)$ is positive this means that we are finding the area between the graph, the x-axis and the lines $x = a$ and $x = b$. It is the shaded area shown in Figure 3.

So we can find the area between a curve and the x-axis by evaluating a definite integral over an interval of the x-axis. Let us take this idea further and pose the problem of finding the volume between a surface and the x,y-plane. Consider a hemisphere of radius a defined by the equation $z = \sqrt{a^2 - x^2 - y^2}$ for $z \geqslant 0$. The projection of the hemisphere onto the x,y-plane is a circle of radius a and is labelled S in Figure 4.

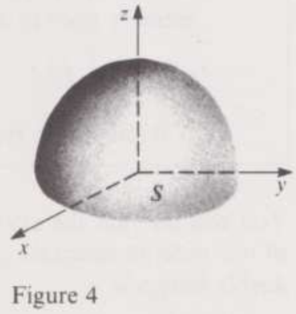


Figure 4

When finding the area between a curve and the x-axis we divided the interval $[a, b]$ (which is the projection of the curve onto the x-axis) into subintervals of width δx_i. In a similar way suppose that we divide the area S into N small area elements so that the ith element has area δA_i and in this element we select a point P_i with co-ordinates (x_i, y_i) (see Figure 5).

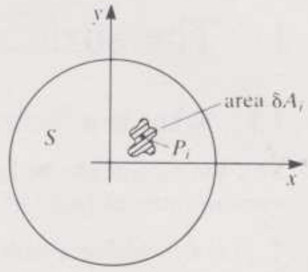


Figure 5

Imagine the ith area element moving perpendicular to the x,y-plane until it reaches the surface of the hemisphere. It will trace out a cylinder, of cross-sectional area δA_i and axis perpendicular to the x,y-plane, between the surface of the hemisphere and the x,y-plane (see Figure 6). Taking the height of this cylinder to be $\sqrt{a^2 - x_i^2 - y_i^2}$, the volume of the cylinder is

$$\sqrt{a^2 - x_i^2 - y_i^2}\,\delta A_i.$$

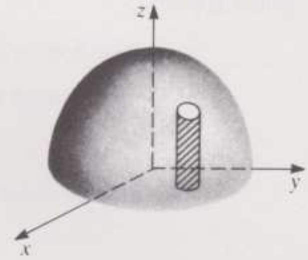


Figure 6

Then we can find an approximate value of the volume of the hemisphere by adding up the volumes of each of the subregions, thus forming the sum

$$\sum_{i=1}^{N} \sqrt{a^2 - x_i^2 - y_i^2}\,\delta A_i.$$

Now if we take the limit of this sum as the number, N, of subregions increases indefinitely and the sizes of the area elements go to zero, we have an expression which is similar to the sum in the definition of a definite integral. So the volume is

$$\lim \sum_{i=1}^{N} \sqrt{a^2 - x_i^2 - y_i^2}\,\delta A_i.$$

The value of this limit, provided that it exists, is an example of a new type of integral called a *surface integral* and we denote it by

$$\int_S \sqrt{a^2 - x^2 - y^2}\,dA.$$

The area S is called the *region of integration* and the integral is called a surface integral because the region of integration is a two-dimensional region (i.e. a surface).

Clearly the limiting process is a complicated one. There are many ways of dividing the area S into elements and many choices of the point P_i in the area element. Under suitable conditions it can be shown that the value of the limit is independent of the method of subdividing the region of integration S.

In the above example we arrived at the idea of a surface integral by thinking about how to calculate a volume. But it is possible to define surface integration more generally without using ideas of area or volume (just as definite integration does not need the idea of area) as follows:

> Suppose that f is a two-dimensional scalar field defined over some region S of the x,y-plane. We divide the region S into N area elements so that the ith element has area δA_i and contains the point P_i (see Figure 7); then we define the **surface integral of f over the region S** to be
>
> $$\lim \sum_{i=1}^{N} f(P_i)\,\delta A_i \qquad (1)$$
>
> where the limit is taken in such a way that the size of each area element goes to zero. We denote this integral by
>
> $$\int_S f\,dA.$$
>
> S is called the **region of integration**.

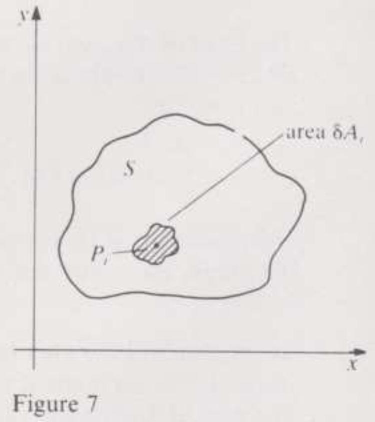


Figure 7

You will see that the evaluation of a surface integral often involves the evaluation of *two* definite integrals and this is why surface integrals are sometimes called **double integrals**.

1.2 Evaluating surface integrals over a rectangular region

Before we can evaluate a surface integral we need to divide the region of integration S into N area elements and there are many different ways of making this subdivision. In this section we shall consider a method of making the subdivision which is convenient when working with the Cartesian co-ordinates x and y. The regions of integration will be areas of the x,y-plane and we write the surface integrals as $\int_S f(x,y)\,dA$, where f is a continuous function of x and y.

We begin by considering the case when the region S is the rectangle formed by the lines $x = a$, $x = b$ and $y = c$, $y = d$, as in Figure 8.

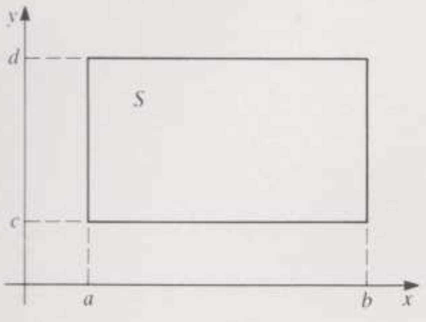


Figure 8

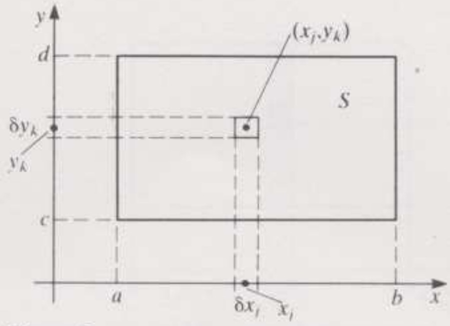


Figure 9

The first step in evaluating the sum (1) in the last subsection is to divide the rectangle into small area elements. The most natural way to do this in Cartesian co-ordinates is:

(i) to divide the interval $[a, b]$ of the x-axis into n subintervals of width $\delta x_1, \delta x_2, \ldots, \delta x_j, \ldots, \delta x_n$ and to choose numbers $x_1, x_2, \ldots, x_n$ such that x_j is in the jth subinterval;

(ii) to divide the interval $[c, d]$ of the y-axis into m subintervals of width $\delta y_1, \delta y_2, \ldots, \delta y_k, \ldots, \delta y_m$ and to choose numbers $y_1, y_2, \ldots, y_m$ such that y_k is in the kth subinterval;

(iii) to choose the area element to be a small rectangle of side lengths δx_j and δy_k. The area of this element is then $\delta x_j \delta y_k$, and it contains the point (x_j, y_k).

These three steps are illustrated in Figure 9.

The rectangle S has been divided into $n \times m$ rectangular area elements. Now we can form the sum $\sum_{i=1}^{N} f(P_i)\,\delta A_i$ by adding the products $f(x_j, y_k)\delta x_j \delta y_k$ over the $n \times m$ rectangular area elements. In terms of the definition in the last subsection, the area δA_i has been replaced by $\delta x_j \delta y_k$ and the surface integral of $f(x, y)$ over the rectangle S is the limit, as the number of area elements increases indefinitely, of the sum of terms $f(x_j, y_k)\,\delta x_j \delta y_k$. Now let us approach this summation in a systematic way by organizing the rectangular area elements into thin strips.

The total number of area elements is
$N = n \times m$.

We can do this in two ways: either we use vertical strips of width $\delta x_1, \delta x_2, \ldots, \delta x_j, \ldots, \delta x_n$ (as in Figure 10) or we use horizontal strips of width $\delta y_1, \delta y_2, \ldots, \delta y_k, \ldots, \delta y_m$ (as in Figure 11).

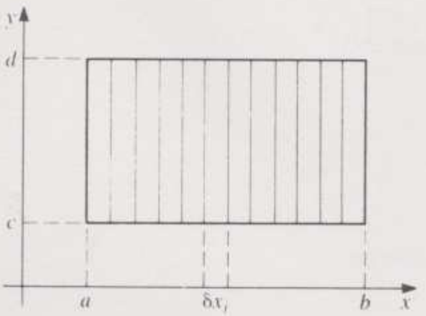


Figure 10

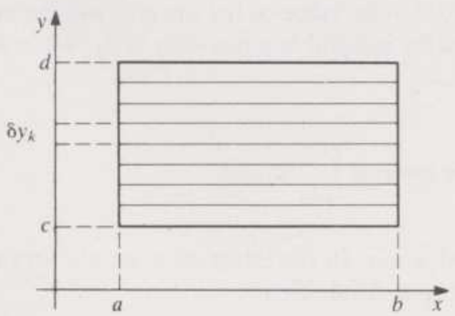


Figure 11

In this subsection we shall consider vertical strips and we shall see that this leads to a definite integral over y followed by a definite integral over x. In the next subsection we shall consider the other ordering.

Consider the jth vertical strip (of width δx_j) shown shaded in Figure 12. We evaluate the contribution to the sum of $f(x_j, y_k)\,\delta x_j\,\delta y_k$ from the m rectangular area elements, like the one shown in this figure, lying in this vertical strip. We can write this contribution as

$$\left[\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right]\delta x_j.$$

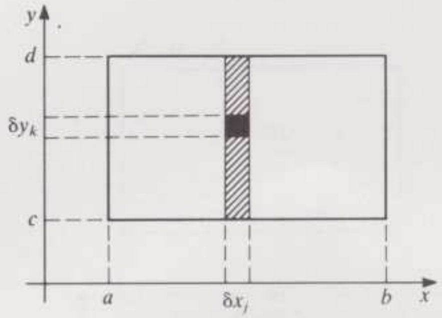


Figure 12

In this summation x_j and δx_j remain unchanged. Now we combine the contributions of the n vertical strips which can be drawn between $x = a$ and $x = b$, and we have an expression involving two summations

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right]\delta x_j.$$

The summation over k is called the inner summation and the summation over j the outer summation. The surface integral of the function f over the region S is the limit of this double summation as the number of rectangles increases indefinitely, i.e. as n and m increase indefinitely.

First let us look at the inner summation enclosed by square brackets. It is like the summation in the last subsection for the definite integral. The sum

$$\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k$$

gives an approximate value for the definite integral

$$\int_{y=c}^{y=d} f(x_j, y)\,dy.$$

In the limit as the lengths δy_k go to zero, the summation equals the definite integral of $f(x_j, y)$ between $y = c$ and $y = d$, i.e. the top and bottom of the jth vertical strip.

We write

$$\lim \sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k = \int_{y=c}^{y=d} f(x_j, y)\,dy.$$

Remember that x_j is a constant number in the summation and is held constant when we evaluate the integral. So the value of the integral will depend on x_j but not on y, that is, the value of the integral is a function of x_j. We will illustrate this with an example.

Example 1

Find the value of the definite integral $\displaystyle\int_{y=1}^{y=2} x_j y\,dy$.

Solution

The function to be integrated is $x_j y$. In the integration we are varying y (between the limits 1 and 2) but keeping x_j fixed. Then

$$\int_{y=1}^{y=2} x_j y\,dy = \left[x_j\frac{y^2}{2}\right]_{y=1}^{y=2} = \tfrac{3}{2}x_j.$$

Exercise 1

Find the value of the definite integral $\int_{y=1}^{y=3} x^2y^3\,dy$, treating x as a constant.

[*Solution on p. 43*]

Suppose that we denote the value of the integral $\int_{y=c}^{y=d} f(x_j, y)\,dy$ by $g(x_j)$; then the outer summation becomes $\sum_{j=1}^{n} g(x_j)\,\delta x_j$. In this summation we are adding over vertical strips between $x = a$ and $x = b$ (see Figure 13).

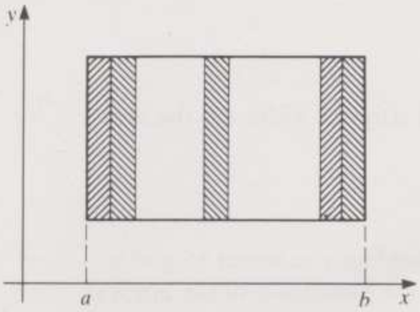


Figure 13

To complete the procedure specified in the definition of a double integral we must make the quantities δx_j go to zero. In the limit the summation approaches the definite integral of $g(x)$ between the limits $x = a$ and $x = b$, i.e.

$$\int_{x=a}^{x=b} g(x)\,dx = \lim \sum_{j=1}^{n} g(x_j)\,\delta x_j.$$

We have expressed the limit of the double summation as δx_j and δy_k go to zero, and therefore the value of the surface integral, in terms of two single integrals, the first over y and the second over x. The result is

$$\int_S f(x,y)\,dA = \int_{x=a}^{x=b} g(x)\,dx \quad \text{where } g(x) = \int_{y=c}^{y=d} f(x,y)\,dy$$

and hence $\int_S f(x,y)\,dA = \int_{x=a}^{x=b}\left[\int_{y=c}^{y=d} f(x,y)\,dy\right]dx$

and each single integral can be evaluated using standard techniques.

Remember that in the integral over y we hold x constant. We could denote this by writing $x = x_j$ in the inner integral but in practice it is not necessary.

Example 2

Find the value of the surface integral of the function $f(x,y) = xy$ over the rectangle bounded by the lines $x = 0$, $x = 3$ and $y = 1$, $y = 2$.

Solution

The region of integration is shown in Figure 14. The surface integral is $\int_S xy\,dA$ and can be written as two single integrals as follows:

$$\int_{x=0}^{x=3}\left[\int_{y=1}^{y=2} xy\,dy\right]dx.$$

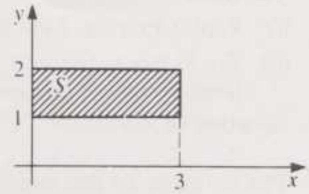


Figure 14

The integral over y, i.e. $\int_{y=1}^{y=2} x_j y\,dy$, was evaluated in Example 1 and we had $\frac{3}{2}x_j$ as the answer. The integral over x becomes

$$\int_{x=0}^{x=3} \frac{3x}{2}\,dx = \left[\frac{3x^2}{4}\right]_0^3 = \frac{27}{4}$$

so that the value of the surface integral of xy over the rectangle is $\frac{27}{4}$.

Exercise 2

Find the value of the surface integral of the function $f(x,y) = x^2y^3$ over the square bounded by the lines $x = 0$, $x = 2$ and $y = 1$, $y = 3$.

[*Solution on p. 43*]

Exercise 3

Show that the value of the surface integral of the function $f(x,y) = xe^{xy}$ over the rectangle bounded by the lines $x = 1$, $x = 3$ and $y = 2$, $y = 3$ is $\left(\frac{e^9}{3} - \frac{e^6}{2} - \frac{e^3}{3} + \frac{e^2}{2}\right)$.

[*Solution on p. 43*]

1.3 A different order of integration

The surface integrals in the previous subsection were evaluated by first integrating over y and then integrating over x; in the summation this is equivalent to summing first over the k's and then the j's. However, we could have used another order for the summations, by first drawing a strip parallel to the x-axis containing y_k and summing over the rectangles of width δx_j for all the j's from 1 to n (see Figure 15).

We would then have

$$\left[\sum_{j=1}^{n} f(x_j, y_k)\,\delta x_j\right]\delta y_k$$

as the first summation. Now summing over all such strips of width δy_k the result would be

$$\int_S f(x,y)\,dA = \int_{y=c}^{y=d}\left[\int_{x=a}^{x=b} f(x,y)\,dx\right]dy.$$

This time in the inner integral we integrate over x holding y constant to give a function of y, then we integrate over y, to complete the evaluation of the integral.

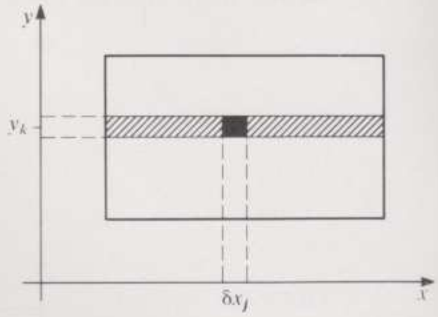


Figure 15

Example 3

Find the value of the double integral in Example 2 but integrate over x first and then over y.

Solution

We have to evaluate the expression

$$\int_{y=1}^{y=2}\left[\int_{x=0}^{x=3} xy\,dx\right]dy.$$

Taking the integral over x first, i.e. $\int_{x=0}^{x=3} xy\,dx$, we treat y as a constant, so we have

$$\int_{x=0}^{x=3} xy\,dx = y\left[\frac{x^2}{2}\right]_{x=0}^{x=3} = \tfrac{9}{2}y.$$

Now the y integration gives

$$\int_{y=1}^{y=2} \tfrac{9}{2}y\,dy = \tfrac{9}{2}\left[\frac{y^2}{2}\right]_{y=1}^{y=2} = \tfrac{27}{4}.$$

As we might have expected, the value of the surface integral is the same as in Example 2; after all, the function and the region of integration are the same. Either order of evaluating the repeated single integrals will give the right answer.

Exercise 4

(i) Repeat Exercise 2 but integrate over x first and then over y.

(ii) Try to repeat Exercise 3 by integrating over x first and then over y. (Do not spend too long on this—if you get into difficulties consult the solution.)

[*Solution on p. 43*]

The solution to this exercise illustrates that one order of integration is sometimes much easier than the other; the solution to Exercise 3 involved more straightforward integrations by integrating over y first and then over x, than by integrating over x first and then over y. So if you get stuck with one order of integration, try a different order. It is advisable to get into the habit of tackling all problems with one ordering (I usually choose vertical strips and integrate over y first and then over x). If one ordering leads to a difficult integral then try the other ordering.

1.4 Evaluating surface integrals over non-rectangular regions

The surface integral of a function over a rectangular region involves two single integrals for which the limits of integration are constants. We will see that for non-rectangular regions we have to be more careful because the strips are no longer of the same length and the limits on the inner integral depend on the variable in the outer integral. To illustrate this consider the following example.

Example 4

Find the value of the surface integral of the function $f(x, y) = xy$ over the region bounded by the curves $y = 2x$, $y = x^2$ and the line $x = 1$.

Solution

We begin by drawing a diagram to show the region of integration; it is the shaded region in Figure 16.

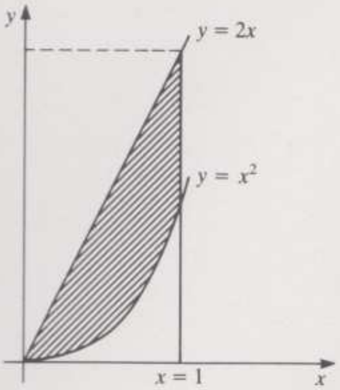


Figure 16

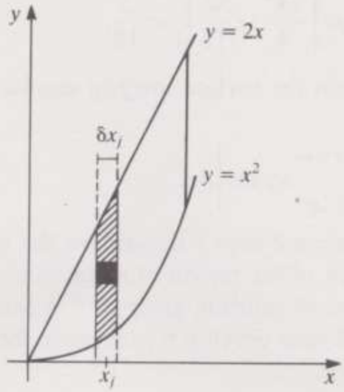


Figure 17

Let us apply the same method of solution as for the rectangular regions in Subsection 1.2 (see Figure 17). The surface integral $\int_S xy\,dA$ can be approximated by the sum

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{m} x_j y_k\,\delta y_k\right]\delta x_j$$

by choosing the rectangular area elements of area $\delta x_j\,\delta y_k$ and organizing the area elements into vertical strips so that the jth strip has width δx_j. When forming the inner summation the area elements occupy positions in the vertical strip between $y_1 = x_j{}^2$ and $y_2 = 2x_j$ (see Figure 18).

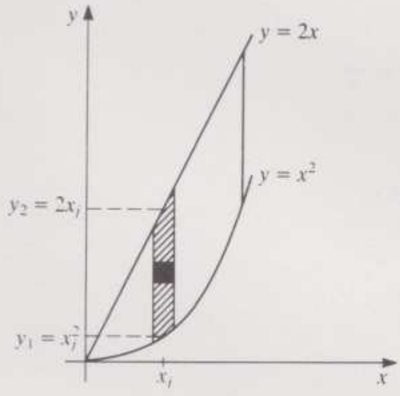


Figure 18

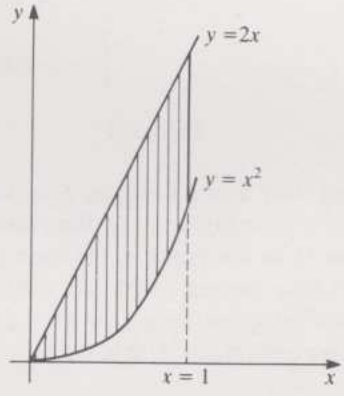


Figure 19

So in the limit as m increases indefinitely the inner summation becomes the definite integral

$$\int_{y=x_j{}^2}^{y=2x_j} x_j y\,dy.$$

We can evaluate this to give

$$\int_{y=x_j{}^2}^{y=2x_j} x_j y\,dy = x_j\left[\frac{y^2}{2}\right]_{y=x_j{}^2}^{y=2x_j} = 2x_j{}^3 - \frac{x_j^5}{2}.$$

The outer summation over j now takes the form

$$\sum_{j=1}^{n}\left(2x_j{}^3 - \frac{x_j^5}{2}\right)\delta x_j.$$

In this summation the vertical strips together fill the space between $x = 0$ and $x = 1$ (see Figure 19). In the limit as n increases indefinitely the summation becomes the definite integral

$$\int_{x=0}^{x=1} \left(2x^3 - \frac{x^5}{2}\right) dx.$$

Evaluating this we have

$$\int_{x=0}^{x=1} \left(2x^3 - \frac{x^5}{2}\right) dx = \left[\frac{2x^4}{4} - \frac{x^6}{12}\right]_0^1 = \frac{5}{12}.$$

We have found that once again the surface integral can be written as two single integrals as follows:

$$\int_S xy\,dA = \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=2x} xy\,dy\right] dx.$$

This time the limits on the integral over y depend on the variable x. You should notice that drawing a diagram of the region of integration is very helpful in getting the limits correct. The method of solution given in this example works for all surface integrals and we shall now develop a procedure for evaluating integrals, by looking at a more general case.

Consider a non-rectangular region S like the one shown in Figure 20. Suppose that a and b are the minimum and maximum values of x, and c and d are the minimum and maximum values of y for the points on the boundary of S. Let $y = \alpha(x)$ and $y = \beta(x)$ be the equations of the boundary curves AB and CD respectively, as shown in Figure 20.

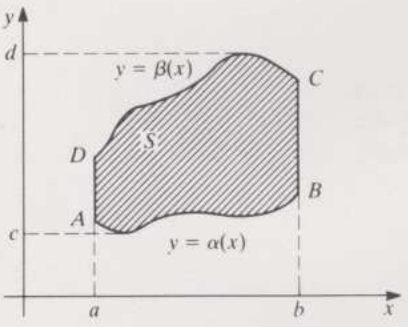


Figure 20

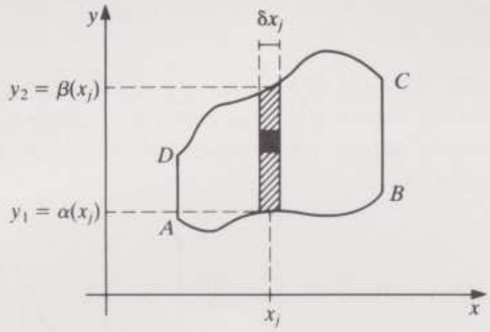


Figure 21

To evaluate the surface integral over S of a function f we want to find the sum of $f(P_i)\,\delta A_i$ over all the area elements contained in S. We choose rectangular area elements of area $\delta x_j\,\delta y_k$ and let P_i be the point in the area element with co-ordinates (x_j, y_k). Consider the area elements within a vertical strip of width δx_j containing the point (x_j, y_k) and lying between $y_1 = \alpha(x_j)$ and $y_2 = \beta(x_j)$, as in Figure 21. If we sum over all the rectangles in this strip we get

$$\left[\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right] \delta x_j.$$

Now summing over the j's as well we get a double summation

$$\sum_{j=1}^{n} \left[\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right] \delta x_j.$$

Consider first the inner summation over k. When forming this summation the area elements occupy positions in the vertical strip between $y_1 = \alpha(x_j)$ and $y_2 = \beta(x_j)$, so in the limit as m increases indefinitely the inner summation becomes the definite integral

$$\int_{y=\alpha(x_j)}^{y=\beta(x_j)} f(x_j, y)\,dy.$$

The value of this single integral depends on x_j not only because $f(x_j, y)$ depends on x_j but also because the limits may depend on x_j. Suppose we denote this

integral by

$$g(x_j) = \int_{y=\alpha(x_j)}^{y=\beta(x_j)} f(x_j, y)\,dy.$$

Then the sum over the j's can be written as

$$\sum_{j=1}^{n} g(x_j)\,\delta x_j.$$

We are summing the terms $g(x_j)$ from $j = 1$ to n; i.e. starting with the first strip at $x = a$ and ending with the nth strip at $x = b$, as in Figure 22. In the limit, as n increases indefinitely, the sum becomes a definite integral with limits $x = a$ and $x = b$ and we write

$$\lim \sum_{j=1}^{n} g(x_j)\,\delta x_j = \int_{x=a}^{x=b} g(x)\,dx$$

$$= \int_{x=a}^{x=b} \left[\int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\,dy \right] dx.$$

The surface integral can be written as two single definite integrals but we must be careful with the limits. The limits on the inner integral are functions of x, not constant numbers.

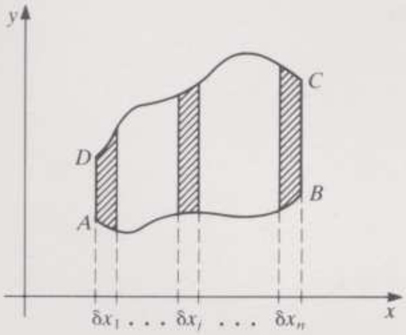


Figure 22

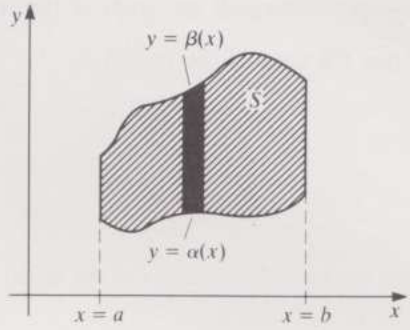


Figure 23

We can summarize these steps into a procedure for evaluating surface integrals using vertical strips.

Procedure 1.4: Evaluating surface integrals

1. Draw a diagram showing the region of integration, S (see Figure 23).
2. Mark on the diagram the minimum value a and maximum value b of x of the points on the boundary. These are the limits of the x integration, i.e. the 'outer' integral.
3. Draw on the diagram a strip parallel to the y-axis and show the lower limit $y = \alpha(x)$ and the upper limit $y = \beta(x)$ of this strip. These are the limits of the y integration, i.e. the 'inner' integral.
4. Write the surface integral as two single integrals, making sure that the outer limits are constant numbers, i.e. the integral becomes

$$\int_S f\,dA = \int_{x=a}^{x=b} \left[\int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\,dy \right] dx.$$

5. Evaluate the inner integral, holding x constant, i.e.

$$g(x) = \int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\,dy.$$

6. Evaluate the outer integral between $x = a$ and $x = b$, i.e.

$$\int_{x=a}^{x=b} g(x)\,dx.$$

Here are some exercises for you to try. Do them by going through the six steps given in Procedure 1.4.

Exercise 5

Find the value of the surface integral of the function $f(x, y) = y$ over the region bounded by the curves $y = x^2$ and $y = x + 2$.

[*Solution on p. 43*]

Exercise 6

Find the value of the surface integral of the function $f(x, y) = x - y$ over the triangle bounded by the lines $y = x - 1$, $x = 3$ and $y = 0$.

[*Solution on p. 43*]

Exercise 7

In Subsection 1.1 we obtained an expression for the volume of a hemisphere of radius a in terms of a surface integral. Evaluate this surface integral.

[*Solution on p. 44*]

Exercise 7 shows that surface integrals can be applied to find volumes. Another application is in the calculation of areas. If we put $f(x, y) = 1$ then the surface integral becomes $\int_S dA$ and we are adding all the areas of the small rectangles drawn within S. This sum is just the area of the region S.

Exercise 8

The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Use a surface integral to calculate the area of the quarter of the ellipse for which $x \geqslant 0$ and $y \geqslant 0$.

[*Solution on p. 44*]

Of course we do not need surface integration to work out this area. Using the techniques of single integration we could draw a strip of width δx as shown in Figure 24 and write the shaded area as

$$b\sqrt{1 - \frac{x^2}{a^2}}\,\delta x$$

so that the required area is

$$\int_{x=0}^{x=a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx.$$

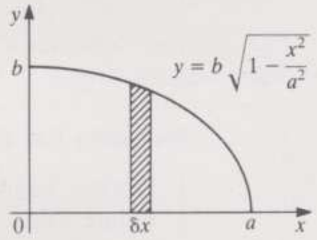


Figure 24

This single integral is what we had in the solution to Exercise 8, after evaluating the inner integral, i.e. after Step 5.

A different ordering for the summations, and hence the two single integrals, results if the area elements are organized into horizontal strips. The following example illustrates how this is done.

Example 5

Find the value of the surface integral of the function xy^2 over the region bounded by the curve $y = x^2$ and the line $y = x$.

Solution

The region of integration is shown in Figure 25. Choose rectangular area elements of area $\delta x_j \delta y_k$ and organize the elements into horizontal strips of width δy_k, as shown in Figure 26.

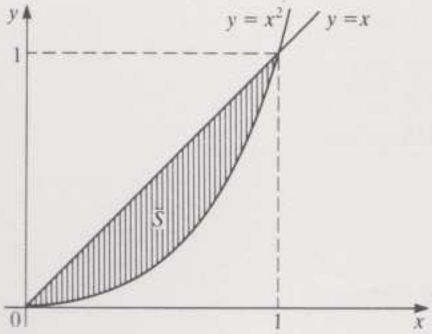


Figure 25

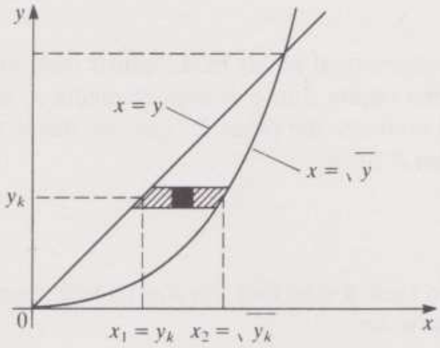


Figure 26

The surface integral can be approximated by sums over j and k, namely

$$\sum_{k=1}^{m}\left[\sum_{j=1}^{n} x_j y_k{}^2\,\delta x_j\right]\delta y_k.$$

When forming the sum over j the rectangular area elements take positions in the shaded strip between $x_1 = y_k$ and $x_2 = \sqrt{y_k}$. The limit of this sum as n increases indefinitely is the integral

$$\int_{x=y_k}^{x=\sqrt{y_k}} x\, y_k{}^2\, dx,$$

which can be evaluated to give

$$\frac{y_k{}^3}{2} - \frac{y_k{}^4}{2}.$$

When forming the sum over k we consider all horizontal strips between $y = 0$ and $y = 1$. The limit of the sum over k as m increases indefinitely is the integral

$$\int_{y=0}^{y=1}\left(\frac{y^3}{2} - \frac{y^4}{2}\right)dy.$$

Evaluating this definite integral gives $\frac{1}{40}$ and this is the value of the surface integral.

Exercise 9

Evaluate $\int_S (x^2 + y^2)\,dA$ where S is the triangle formed by the lines $y = 0$, $y = x - 1$ and $x = 2$, by

(i) integrating over y first and then over x;

(ii) integrating over x first and then over y.

[*Solution on p. 44*]

Either order of integration, i.e. x first or y first, is acceptable although the calculations are often easier for one particular order. It is important to be careful with the limits on the two single integrals. The limits on the outer integral are constant numbers whereas the limits on the inner integral will in general be non-constant functions. Only for rectangular regions of integration will the four limits be constant numbers.

When using Cartesian co-ordinates, we often omit the brackets and limits and write a surface integral as

$$\iint_S f(x, y)\,dx\,dy \qquad \text{or} \qquad \int_S f(x, y)\,dA.$$

Of course we need the limits and an ordering of the two single integrals when the surface integral is to be evaluated.

Summary of Section 1

1. Suppose that f is a two-dimensional scalar field defined over some region S of the x,y-plane. We divide the region S into N area elements so that the ith element has area δA_i and contains the point P_i; then we define **the surface integral of f over the region S** to be

$$\lim \sum_{i=1}^{N} f(P_i)\,\delta A_i$$

where the limit is taken in such a way that the size of each area element goes to zero. We denote this integral by

$$\int_S f\,dA.$$

S is called the **region of integration**.

2. In Cartesian co-ordinates, if the region of integration S is the rectangle bounded by the lines $x = a$, $x = b$ and $y = c$, $y = d$, then the surface integral $\int_S f(x, y)\,dA$ can be found by evaluating two single integrals written as

$$\int_{x=a}^{x=b}\left[\int_{y=c}^{y=d} f(x,y)\,dy\right]dx,$$

or alternatively by evaluating

$$\int_{y=c}^{y=d}\left[\int_{x=a}^{x=b} f(x,y)\,dx\right]dy.$$

3. If the region S is non-rectangular in shape then we use Procedure 1.4 for evaluating the surface integral $\int_S f(x, y)\,dA$.

4. If Procedure 1.4 gives unwieldy integrals we may be more successful integrating over x first, and then integrating over y.

2 Multiple integrals useful in mechanics

The first section of this unit introduced the surface integral of a function of two variables and developed a procedure for evaluating such integrals using the Cartesian co-ordinates x and y when the region of integration is an area in the x,y-plane. Section 1, then, dealt just with the definition and evaluation of one type of multiple integral, the surface integral.

In this section we use multiple integrals to define two physical properties of an object, which will be taken up in the next mechanics unit.

2.1 Volume integrals in mechanics (Television Subsection)

A **volume integral** is the integral of a scalar function of three variables over a region of three-dimensional space. The television programme uses the volume integral to define the mass of a sphere and a cylinder. To illustrate how multiple integrals might be useful in mechanics we start by investigating the mass of a flat plate with non-uniform density and expressing the mass as a surface integral. This will set the scene for some of the ideas in the programme.

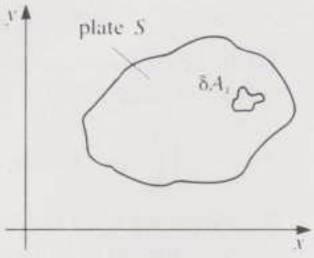


Figure 1

Consider a flat plate S of area A, constant thickness h and density ρ which depends on position. Suppose that we split the plate into N small elements so that the ith element has surface area δA_i and hence volume $h\,\delta A_i$ (see Figure 1).

The mass of the element is approximately given by

$$\delta M_i \simeq \rho_i\,\delta A_i\,h,$$

where ρ_i is the value of the density taken at some point inside the ith element.

The total mass of the plate is the sum

$$\sum_{i=1}^{N} \delta M_i \simeq \sum_{i=1}^{N} h\rho_i \delta A_i = h\sum_{i=1}^{N} \rho_i \delta A_i.$$

As we take smaller and smaller elements the change in ρ throughout each element is reduced and the sum becomes a closer approximation to the mass of the plate. In the limit as each element shrinks to a point the summation $\sum_{i=1}^{N} \rho_i \delta A_i$ becomes the surface integral of the function ρ over the region S. We have

$$\text{the mass of the plate} = h\int_S \rho(x,y)\,dA.$$

Here then is an example of a surface integral being used to evaluate the physical property of the plate called the mass. So for various shapes we can calculate the mass by evaluating surface integrals, if we know $\rho(x,y)$.

In the special case when the density of the material is constant throughout the plate then we have

$$\text{mass} = h\int_S \rho\,dA = h\rho\int_S dA = h\rho A.$$

For example, a uniform solid cylinder of radius a and height h can be thought of as a thick plate of thickness h. The cross-section is a circle of radius a so that the cross-sectional area is πa^2. The mass of the cylinder is then $\pi a^2 h\rho$.

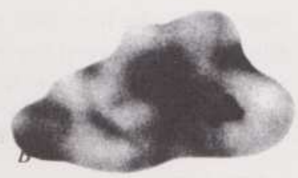


Figure 2

To extend these ideas to volume integrals let us now consider a body B of non-uniform density ρ, which does not have a constant cross-sectional area. Suppose we split the volume into N small 'chunks' which we shall call volume elements, so that the ith 'chunk' has volume δV_i (see Figure 2). Then the mass of the ith element, δM_i, is given approximately in terms of δV_i by

$$\delta M_i \simeq \rho_i \delta V_i$$

where ρ_i is the density of the material evaluated at a point (x_i, y_i, z_i) in the volume element. The total mass of the body is given by the sum

$$M = \sum_{i=1}^{N} \delta M_i \simeq \sum_{i=1}^{N} \rho_i \delta V_i.$$

If we take volume elements of smaller and smaller size then in the limit the summation can be written in the form of an integral:

$$M = \int_B \rho(x,y,z)\,dV.$$

This is an example of a volume integral.

Again if the density is constant this can be simplified to give

$$M = \rho\int_B dV = \rho V,$$

where V is the volume of the body B.

In the television programme V is used to represent both the body (which we have labelled here by B) and its volume.

Exercise 1

(i) In Section 1 (Exercise 7) we showed that the volume of a hemisphere of radius a is $\frac{2}{3}\pi a^3$. What is the mass of a sphere of constant density and radius a?

(ii) A hollow sphere of constant density has internal radius b and external radius a. Find the mass of the hollow sphere.

[*Solution on p. 45*]

The television programme can be summarized as follows.

Part 1: The programme begins with two experiments showing different motions of two cylinders. When placed in trolleys that slide down (cycloidal) tracks their motions are indistinguishable. The cylinders look identical and they have the same mass. The cylinders are then *rolled* down the tracks and we find that one cylinder

goes faster than the other. So in rotational motion there is a difference between the cylinders. An important property of rigid bodies, called the *moment of inertia*, is then defined in terms of a sum.

Part 2: We show how to choose a shape for the volume elements and how to organize them in a systematic way for evaluating the sum.

Part 3: We show that in the limit as the size of the volume elements gets smaller and smaller (and hence the number of elements increases) the summation can be written as a volume integral.

Part 4: To evaluate the masses and moments of inertia of the cylinders we introduce a new co-ordinate system called cylindrical polar co-ordinates. We find expressions for the masses and moments of inertia of the cylinders and show that the moments of inertia differ; we can thus explain the difference in the rotational motions of the bodies by looking at their moments of inertia.

Part 5: We introduce spherical polar co-ordinates to evaluate the masses and moments of inertia of two indentical-looking spheres. We obtain expressions for their masses and their moments of inertia. Again the moments of inertia differ so we predict that they will roll with different speeds on the cycloidal track.

Now watch the television programme '*Multiple Integrals*' to find out what the difference is between the two sets of spheres and cylinders and what does happen when the spheres are rolled.

TV27

Here is the definition of moment of inertia given in the television programme:

> For a body B we define the **moment of inertia of the body about a fixed axis** (see Figure 3) by
>
> $$I = \lim \sum_{i=1}^{N} \delta M_i d_i^2$$
>
> where δM_i is the mass of the ith volume element and d_i is the distance of the ith volume element from the axis. In terms of a volume integral the limit of this sum is
>
> $$I = \int_B (\rho d^2)\, dV.$$

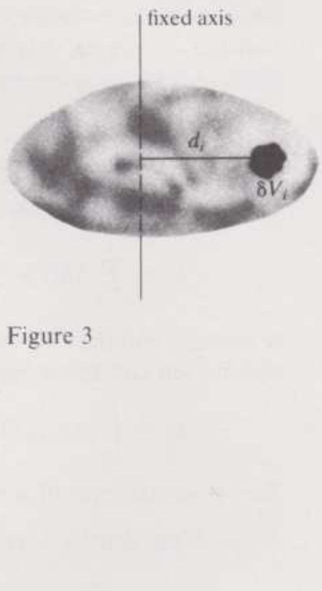


Figure 3

The television programme introduces new ideas which are followed up in Sections 3 and 4 of this unit and in the next mechanics unit. Section 3 develops a procedure for evaluating volume integrals in Cartesian co-ordinates by subdivision into rectangular blocks. Section 4 defines cylindrical and spherical polar co-ordinates and uses them to evaluate volume integrals when the region of integration has axial or spherical symmetry. The theory of the rotation of bodies is introduced in the next mechanics unit, and the definition and explanation of moments of inertia are given there.

The secret of the cylinders and spheres is that one of each is hollow and one of each is solid. The mass of each cylinder is the same, which means that the hollow cylinder has most of its mass concentrated on the edge, and so the moments of inertia differ. For the hollow cylinder the moment of inertia is $\frac{1}{2}M(a^2 + b^2)$ and for the solid cylinder it is $\frac{1}{2}Ma^2$. Because $a^2 + b^2 > a^2$, the moment of inertia of the hollow cylinder is greater than that of the solid cylinder, so that the hollow cylinder travels more slowly down the track than does the solid cylinder.

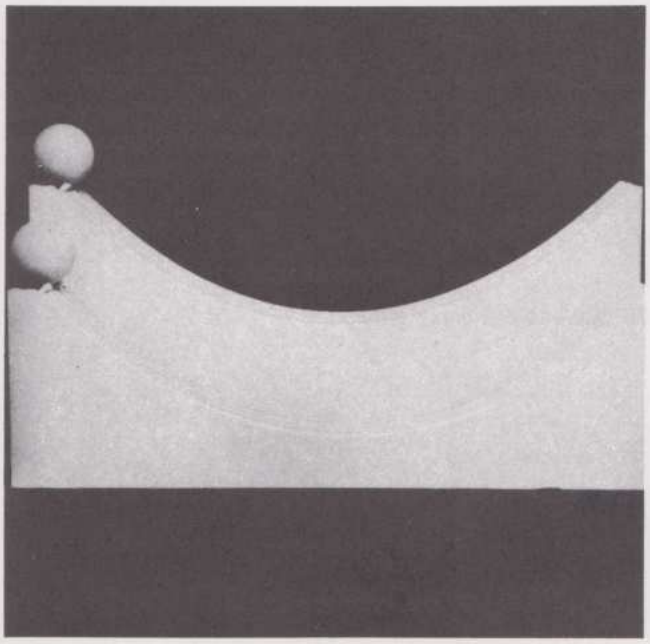
a is the external radius and b is the internal radius.

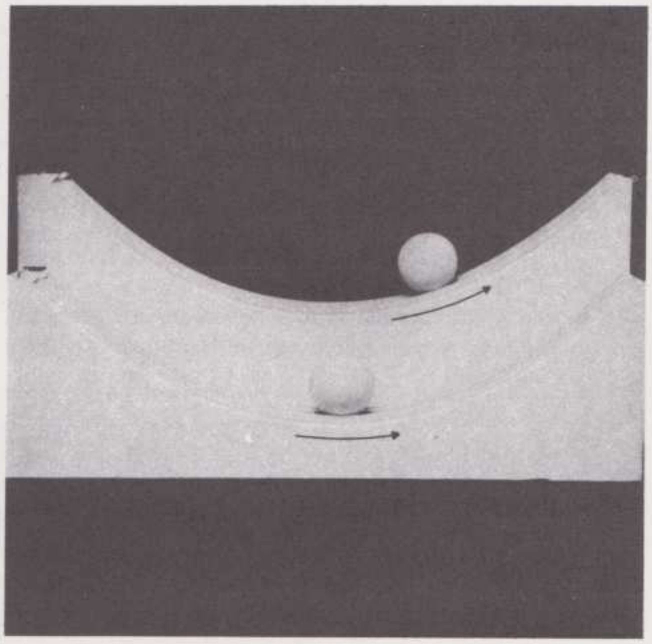
For the spheres we have the following formulae from the programme:

	mass	moment of inertia
hollow sphere	$M = \frac{4}{3}\pi\rho_1(a^3 - b^3)$	$I_1 = \frac{8}{15}\pi\rho_1(a^5 - b^5)$
solid sphere	$M = \frac{4}{3}\pi\rho_2 a^3$	$I_2 = \frac{8}{15}\pi\rho_2 a^5$

Although the densities of the spheres are different, their masses are the same.

The television programme ended with the two spheres rolling on the cycloidal track. The programme shows that the sphere on the top track travels faster. Is this the solid sphere or the hollow sphere? The following exercise helps you to answer that question.





Exercise 2

(i) Using the formulae in the table above find the values of $\dfrac{I_1}{M}$ and $\dfrac{I_2}{M}$.

(ii) Using the result in (i), which sphere has the greatest moment of inertia?

(iii) Hence answer the question: is the faster sphere hollow or solid?

[*Solution on p. 45*]

2.2 Changing co-ordinates in surface integrals

In the television programme we evaluated volume integrals over cylinders and spheres in co-ordinate systems based on the geometry of regions of integration; for instance we used spherical polar co-ordinates to find the moment of inertia of a sphere. In Section 4 of this unit we will expand on these ideas.

In evaluating surface integrals it is often convenient to choose a non-Cartesian co-ordinate system. For the remainder of this section we will develop a procedure for using plane polar co-ordinates in surface integrals. The steps in the procedure carry over to volume integrals in three dimensions but it is sometimes easier to visualize what is going on in two dimensions.

To set the scene for the tape subsection that follows, evaluate the surface integral in the following exercise using Cartesian co-ordinates.

Exercise 3

Find the moment of inertia of a disc of constant density ρ, radius a and thickness h about an axis perpendicular to the plane of the disc and passing through its centre.

[*Solution on p. 45*]

Using Cartesian co-ordinates in this exercise gives an answer but the x integration is not very pleasant. Now consider the following example:

Example 1

Evaluate the surface integral of the function $e^{x^2+y^2}$ over the quarter circle of radius 1 bounded by the lines $x = 0$ and $y = 0$.

Discussion

First let us see what we get when using Cartesian co-ordinates. Using Procedure 1.4 we can write the surface integral in terms of single integrals in the following way (see Figure 4):

$$\int_S e^{x^2+y^2}\,dA = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=\sqrt{1-x^2}} e^{x^2+y^2}\,dy\right]dx = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=\sqrt{1-x^2}} e^{x^2}e^{y^2}\,dy\right]dx.$$

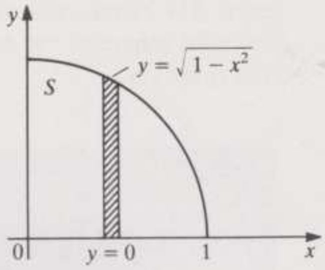


Figure 4

Now we are stuck because we do not know how to evaluate the integral of e^{y^2}.

This exercise and example suggest that there are two reasons for changing co-ordinates; first, the region of integration may have some special symmetry that can be exploited (e.g. in these examples perhaps we can use the fact that the regions of integration are parts of circles); and second, we may not be able to integrate the function itself. Of course a change of co-ordinates may still not help with the second problem.

A co-ordinate system that is based on the symmetry of a circle is the *plane polar co-ordinate system* which we remind you of below.

> Any point P can be represented uniquely by specifying the distance OP ($=r$, say) and the angle, θ, that OP makes with the x-axis (see Figure 5). Then (r, θ) are called the **plane polar co-ordinates** of P. The algebraic relation of Cartesian co-ordinates to plane polar coordinates is given by
>
> $$x = r\cos\theta,$$
>
> $$y = r\sin\theta.$$

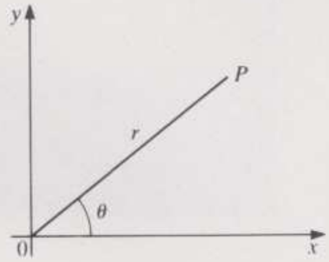


Figure 5

This co-ordinate system turns out to be very useful when evaluating surface integrals where the region of integration is all or part of a circle. The audio-tape for this unit develops an algorithm for evaluating surface integrals using plane polar co-ordinates.

2.3 Using plane polar co-ordinates in surface integrals (Tape Subsection)

Read the problems in Frames (1a) *and* (1b) *and start the tape when you are ready.*

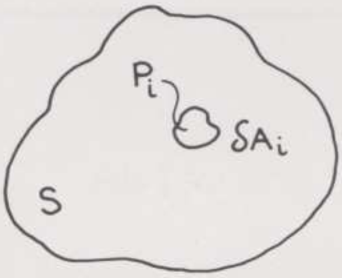
1a **Problem 1**

Evaluate the surface integral of the function x^2+y^2 over a circle of radius a and centre at the origin.

1b **Problem 2**

Evaluate the surface integral of the function $e^{x^2+y^2}$ over the region bounded by $x^2+y^2 \leqslant a^2$, $x \geqslant 0$ and $y \geqslant 0$

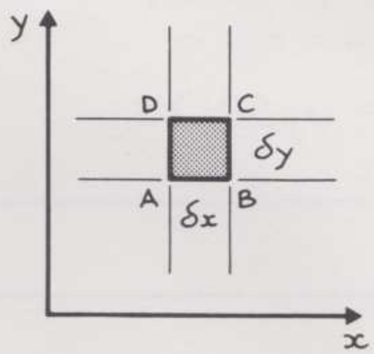
2 **surface integral**



$$\int_S f \, dA = \lim \sum_{i=1}^{N} f(P_i)\,\delta A_i$$

In polar coordinates, what is δA?

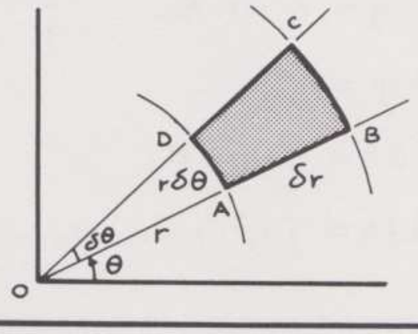
3 **in Cartesian coordinates**



shape is a rectangle

$$\delta A = \delta x\,\delta y$$

4 **in polar coordinates**



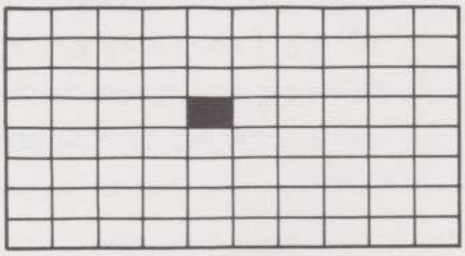
r = constant are arcs of circles

θ = constant are spokes

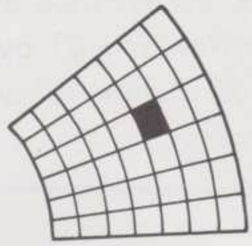
$$\delta A = (r\,\delta\theta) \times \delta r$$

5

Cartesian coordinates

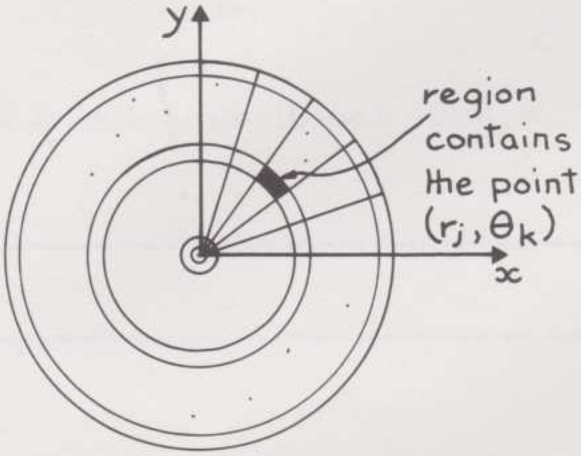


polar coordinates



6

problem 1



$$\int_S (x^2 + y^2)\, dA$$

divide region S into N area elements

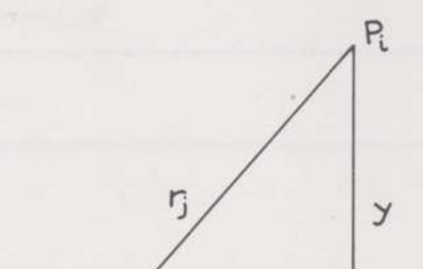
m circles

n spokes

$N = m \times n$

$$\delta A = (r_j\, \delta\theta_k)\, \delta r_j$$

7



in polar coordinates

$$x = r_j \cos \theta_k$$

$$y = r_j \sin \theta_k$$

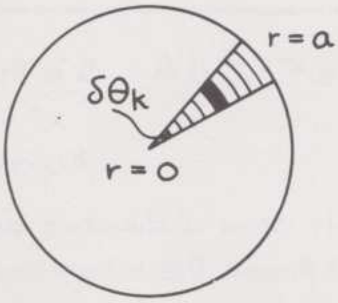
$$x^2 + y^2 = r_j^2$$

$$\delta A = r_j\, \delta r_j\, \delta\theta_k$$

$$\int_S (x^2 + y^2)\, dA \simeq \sum r_j^2 \cdot r_j\, \delta r_j\, \delta\theta_k$$

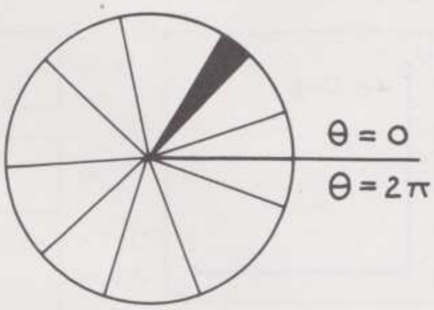
8 Limits

sum over j's $\left(\sum_{j=1}^{n} r_j^3 \delta r_j\right) \delta\theta_k$



$$\left.\begin{array}{l} r = 0 \\ r = a \end{array}\right\} \text{limits on } r$$

sum over k's $\sum_{k=1}^{m}\left(\sum_{j=1}^{n} r_j^3 \delta r_j\right) \delta\theta_k$



$$\left.\begin{array}{l} \theta = 0 \\ \theta = 2\pi \end{array}\right\} \text{limits on } \theta$$

9 two single integrals

$$\int_S r^2\, dA = \lim \sum_{k=1}^{m}\left(\sum_{j=1}^{n} r_j^3 \delta r_j\right) \delta\theta_k = \int_{\theta=0}^{\theta=2\pi} \left[\int_{r=0}^{r=a} r^3\, dr\right] d\theta$$

10 evaluation

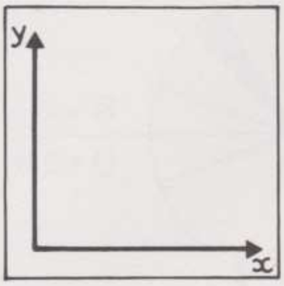
$$\int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=a} r^3\, dr\right) d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\frac{a^4}{4}\right) d\theta = \frac{a^4}{4} 2\pi = \frac{\pi a^4}{2}$$

problem 2

cover up these frames

11 $\int_S e^{x^2+y^2}\, dA$ S is the region $x^2+y^2 \leqslant a^2$, $x \geqslant 0$ and $y \geqslant 0$

Step 1: draw a diagram and write x, y, δA and the integral in polar coordinates



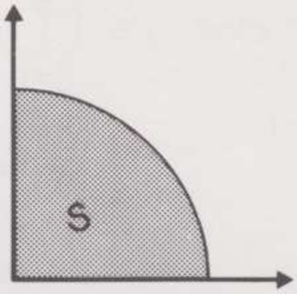
$x =$ ☐

$y =$ ☐

$\delta A =$ ☐

$$\int_S e^{x^2+y^2}\, dA = \boxed{\int\left[\int \qquad dr\right]d\theta}$$

11a



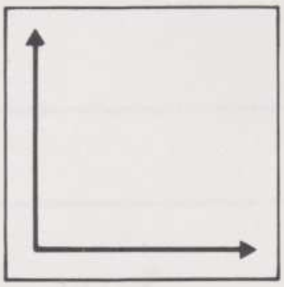
$x = r\cos\theta$

$y = r\sin\theta$

$\delta A = r\,\delta r\,\delta\theta$

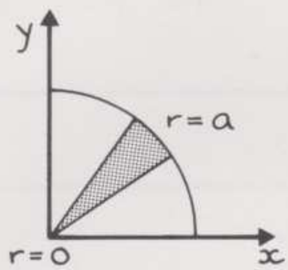
$$\int\left[\int e^{r^2} r\, dr\right]d\theta$$

12 Step 2: draw in sector to show r limits



r limits $\begin{cases} r = \square \\ r = \square \end{cases}$

12a

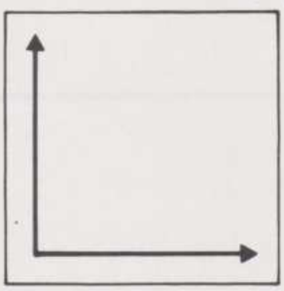


r limits:

$r = 0$

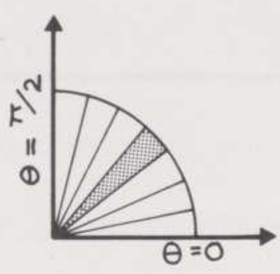
$r = a$

13 Step 3: draw in bounding lines to show θ limits



θ limits $\begin{cases} \theta = \square \\ \theta = \square \end{cases}$

13a



θ limits:

$\theta = 0$

$\theta = \frac{\pi}{2}$

14 Step 4: write the surface integral as two single integrals including the limits of integration

$\boxed{\quad\quad\quad\quad}$

14a

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}}\left(\int_{r=0}^{r=a} e^{r^2} r\,dr\right)d\theta$$

15 Step 5: evaluate inner integral

$$\int_{r=0}^{r=a} e^{r^2} r\,dr = \boxed{\quad\quad}$$

Step 6: evaluate outer integral

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \boxed{\quad\quad}\, d\theta = \boxed{\quad\quad}$$

15a

$$\int e^{r^2} r\,dr = \frac{1}{2}e^{r^2} + c$$

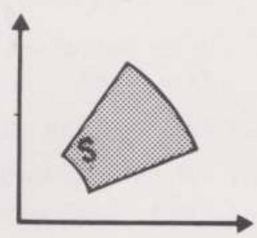
$$\int_{r=0}^{r=a} e^{r^2} r\,dr = \frac{1}{2}\left(e^{a^2}-1\right)$$

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2}\left(e^{a^2}-1\right)d\theta = \frac{\pi}{4}\left(e^{a^2}-1\right)$$

16 using polar coordinates in surface integrals

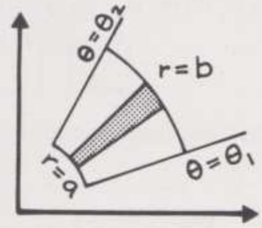
Step 1: draw a diagram showing the region of integration and write the surface integral in terms of r, θ

$$\delta A = r\,\delta r\,\delta\theta$$



Step 2: draw in a sector to show r limits

Step 3: draw in bounding lines to show the θ limits



Step 4: write the surface integral as two single integrals

$$\int_{\theta=\theta_1}^{\theta=\theta_2}\left(\int_{r=a}^{r=b} f(r,\theta)\,r\,dr\right)d\theta$$

Step 5: evaluate the inner integral $\int_{r=a}^{r=b}$

Step 6: evaluate the outer integral $\int_{\theta=\theta_1}^{\theta=\theta_2}$

Summary of Section 2

1. Suppose that f is a three-dimensional scalar field whose domain includes a region B. The **volume integral** of f over the region B is defined as

$$\int_B f\,dV = \lim \sum_{i=1}^{N} f(P_i)\,\delta V_i$$

where B is divided into a variable number, N, of volume elements, the ith of which has volume δV_i and contains the point P_i, and the limit is taken in such a way that all the volume elements become very small. We call B the **region of integration.**

2. If ρ is the scalar field giving the local density of a body occupying a region B, then the **mass** of the body is

$$\int_B \rho\,dV.$$

The **moment of inertia** of this same body **about some given axis** is

$$\int_B \rho\,d^2\,dV$$

where d denotes distance from this axis.

3. The mass of a hollow sphere of uniform density ρ with external radius a and internal radius b is $\frac{4\pi}{3}\rho(a^3 - b^3)$ and its moment of inertia about an axis through its centre is $\frac{8\pi}{15}\rho(a^5 - b^5)$. The corresponding formulae for solid spheres are obtained by setting $b = 0$.

4. To evaluate a surface integral $\int_S f\,dA$ using polar co-ordinates the following procedure can be used.

   (i) Draw a diagram showing the region of integration, S, and its bounding lines.

   (ii) Write the surface integral in terms of r and θ using the formulae

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$\delta A = r\,\delta r\,\delta\theta.$$

   In the limit of very small area elements this last formula becomes $dA = r\,dr\,d\theta$.

   (iii) Find the minimum and maximum values of θ in S, say θ_1 and θ_2.

   (iv) For a strip of fixed θ indicate the minimum and maximum values of r, say a and b (here assumed to be independent of θ, for simplicity).

   (v) Write the surface integral in terms of two single integrals

$$\int_{\theta=\theta_1}^{\theta=\theta_2} \left[\int_{r=a}^{r=b} f(r,\theta)\,r\,dr \right] d\theta.$$

   (vi) Evaluate the inner integral, then the outer integral.

3 The volume integral

3.1 The volume integral defined

The television programme showed applications of the volume integral in mechanics, and in particular we saw that the mass and moment of inertia can be expressed as volume integrals. In this section and the next section we shall return to the theory of volume integrals and leave the applications until later units.

Let us begin by defining the volume integral in a similar way to the definition we used for the surface integral in Section 1.

Suppose that a scalar field f is defined for all points within a region B of three-dimensional space. If we divide the region B into N subregions, called **volume elements**, so that the ith volume element has volume δV_i and contains the point P_i, then we define the **volume integral of f over the region B** to be

$$\lim \sum_{i=1}^{N} f(P_i)\,\delta V_i$$

where the limit is taken in such a way that the size of each volume element goes to zero. This integral is denoted by

$$\int_B f\,dV.$$

B is called the **region of integration**.

Before we can evaluate a volume integral we must choose a shape for the volume elements and there are many different ways of doing this. In this section we shall consider a method of making the subdivision which is convenient when working in Cartesian co-ordinates. We shall see that the evaluation of a volume integral is just an extension of the method for evaluating surface integrals, except that now we shall have *three* single integrals to evaluate instead of two. For this reason volume integrals are sometimes called **triple integrals**.

3.2 Evaluating volume integrals over rectangular regions

In this subsection we shall consider volume integrals over a cuboid or a rectangular block whose faces are parallel to the planes defining the Cartesian co-ordinate system. The method of evaluation of volume integrals over such regions is very similar to that of evaluating surface integrals over a rectangle in the x,y-plane. The method is best illustrated by an example.

Example 1

Find the value of the volume integral of the function $f(x, y, z) = x^2 + y^2 + z^2$ over the rectangular block bounded by the planes $x = 0$, $x = 1$, $y = 2$, $y = 4$, $z = 1$ and $z = 3$.

Solution

The region of integration is shown in Figure 1.

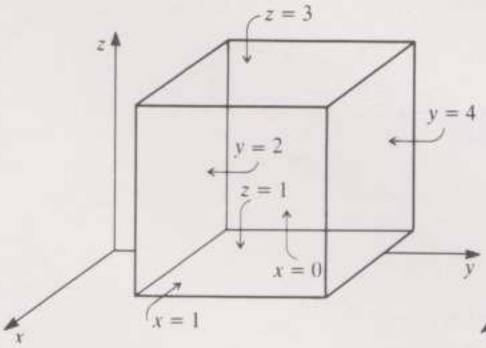


Figure 1

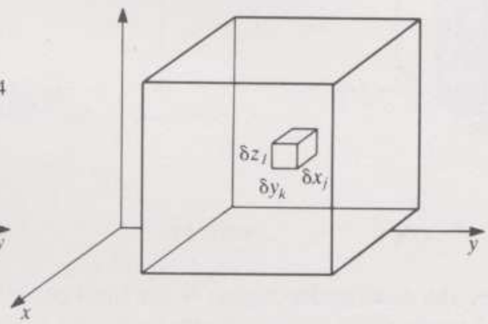


Figure 2

For the volume elements we choose $n \times m \times r$ small rectangular blocks, where the ith block has side lengths δx_j, δy_k, δz_l so that the volume of the ith element (see Figure 2) is

$$\delta V_i = \delta x_j\,\delta y_k\,\delta z_l.$$

The total number of volume elements is $N = n \times m \times r$.

We let P_i, with co-ordinates (x_j, y_k, z_l) be a point in the ith block and we evaluate the sum $\sum_{i=1}^{N} f(P_i)\,\delta V_i$ in a systematic way as follows.

First consider a vertical column between the faces $z = 1$ and $z = 3$ as shown in Figure 3. For this column x_j and y_k are constants. Then we evaluate the sum of

$$(x_j^2 + y_k^2 + z_l^2)\,\delta x_j\,\delta y_k\,\delta z_l$$

for the r blocks within this vertical column between $z = 1$ and $z = 3$.

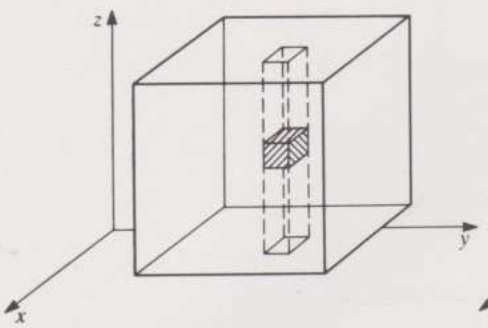


Figure 3

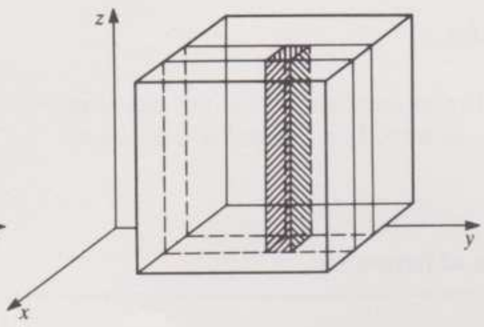


Figure 4

This sum can be written as

$$\left[\sum_{l=1}^{r} (x_j^2 + y_k^2 + z_l^2)\,\delta z_l\right]\delta x_j\,\delta y_k. \qquad (1)$$

In this summation x_j and y_k remain unchanged.

The next step is to consider all the vertical columns that are contained within a *slice* for which x_j is constant (see Figure 4). We evaluate the sum of the expression (1) for the m columns between $y = 2$ and $y = 4$ to give

$$\left(\sum_{k=1}^{m}\left[\sum_{l=1}^{r} (x_j^2 + y_k^2 + z_l^2)\,\delta z_l\right]\delta y_k\right)\delta x_j.$$

Finally we sum over the n slices between $x = 0$ and $x = 1$ (see Figure 5) to give an expression involving three summations:

$$\sum_{i=1}^{N} f(P_i)\,\delta V_i = \sum_{j=1}^{n}\left(\sum_{k=1}^{m}\left[\sum_{l=1}^{r} (x_j^2 + y_k^2 + z_l^2)\,\delta z_l\right]\delta y_k\right)\delta x_j. \qquad (2)$$

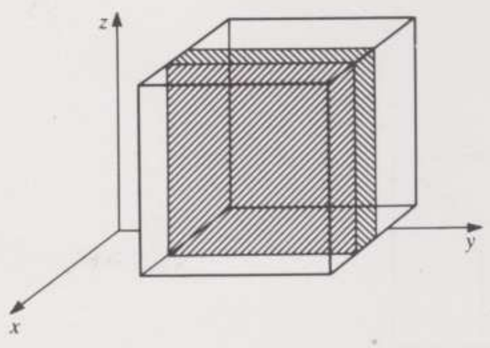


Figure 5

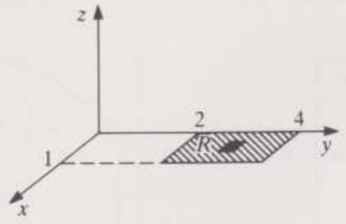


Figure 6

The volume integral of f over the rectangular region is the limit of this triple summation as the number of rectangular blocks of volume $\delta x_j\,\delta y_k\,\delta z_l$ increases indefinitely, i.e. as n, m and r increase indefinitely.

Now look at expression (2) and consider the inner summation over l. The inner summation as r increases indefinitely becomes the definite integral between $z = 1$ and $z = 3$, i.e. the end faces of the vertical column. The definite integral is

$$\int_1^3 (x_j^2 + y_k^2 + z^2)\,dz.$$

In this integral the values of x_j and y_k remain unchanged; they are the co-ordinates of the centre of the rectangle forming the projection of the volume element onto the x,y-plane.

We have

$$\int_1^3 (x_j^2 + y_k^2 + z^2)\,dz = \left[(x_j^2 + y_k^2)z + \frac{z^3}{3}\right]_1^3$$
$$= 2(x_j^2 + y_k^2) + \frac{26}{3}.$$

We are now left with the double summation

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{m}\left(2(x_j^2 + y_k^2) + \frac{26}{3}\right)\delta y_k\right]\delta x_j.$$

The region over which we are forming this sum is the rectangle R bounded by the lines $x = 0$, $x = 1$, $y = 2$ and $y = 4$, and this is the projection of the rectangular block onto the x,y-plane (see Figure 6). The vertical column becomes a rectangular area element in this projection. So this sum is an approximate value for the surface integral of the function g defined by

$$g(x, y) = 2(x^2 + y^2) + \frac{26}{3}$$

over the rectangle R. Now we can evaluate such surface integrals using the techniques of Section 1. We have

$$\int_R g(x, y)\,dA = \int_{x=0}^{x=1}\left[\int_{y=2}^{y=4}\left(2(x^2 + y^2) + \frac{26}{3}\right)dy\right]dx$$
$$= \int_{x=0}^{x=1}\left(4x^2 + \frac{164}{3}\right)dx$$
$$= 56.$$

In this example we reduced the volume integral over a block B to a double integral over the region R which is the projection of B onto the x,y-plane. The systematic approach of building the volume elements (the blocks) into columns, then the columns into slices and finally the slices into the region B was illustrated on the television programme and compared with cutting up a loaf of bread. We took a *sliced* loaf, cut a slice into '*soldiers*' and the soldiers into *small mouthfuls.* These small mouthfuls are the volume elements.

In the example we have 'built up' the region B by taking vertical columns (i.e. with an axis parallel to the z-axis) and slices parallel to the y,z-plane. Other choices are possible. For instance, if we form columns as in Figure 7 with an axis parallel to the x-axis then the first single integral would be

$$h(y, z) = \int_{x=0}^{x=1} (x^2 + y^2 + z^2)\,dx = \frac{1}{3} + (y^2 + z^2)$$

and the resulting surface integral would be of the function $h(y, z)$ over the rectangle bounded by the lines $y = 2$, $y = 4$, $z = 1$ and $z = 3$ (see Figure 8).

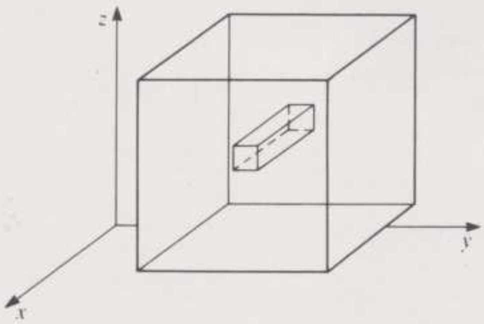


Figure 7

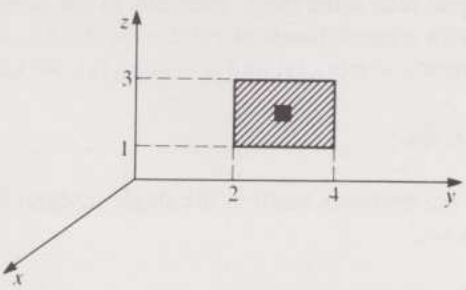


Figure 8

If we chose this way of forming the summation $\sum_{i=1}^{N} f(P_i)\,\delta V_i$ we would get the surface integral

$$\int_{y=2}^{y=4}\left[\int_{z=1}^{z=3}\left(\frac{1}{3}+(y^2+z^2)\right)dz\right]dy$$
$$=\int_{y=2}^{y=4}\left[2\left(\frac{1}{3}+y^2\right)+\frac{26}{3}\right]dy$$
$$=56.$$

So either order of forming the volume elements into the total volume gives the same answer.

Exercise 1

There is a third way of 'building' the blocks into a column and this leads to a surface integral over a rectangle in the x,z-plane. Describe this order and check that it gives the same answer as before.

[*Solution on p. 46*]

Exercise 2

Find the moment of inertia (as defined in Section 2) of the cube of constant density ρ bounded by the planes $x=-1$, $x=1$, $y=-1$, $y=1$, $z=-1$ and $z=1$, about the x-axis. Show that this moment of inertia is $\frac{2}{3}$ of the mass of the cube.

[*Solution on p. 46*]

Exercise 3

The density of a rectangular block B bounded by the planes $x=1$, $x=2$, $y=0$, $y=3$, $z=-1$ and $z=0$ is given by the function $\rho(x,y,z)=x(y+1)-z$. Find the mass of the block.

[*Solution on p. 46*]

3.3 Evaluating volume integrals over non-rectangular regions

If the region of integration B is not a rectangular block then the limits on the first two integrals will be functions and not constants. Finding the limits of integration can be tricky. Figure 9 shows a region B bounded above by the surface $z=\psi(x,y)$ and bounded below by the surface $z=\phi(x,y)$. Its projection onto the x,y-plane is denoted by S.

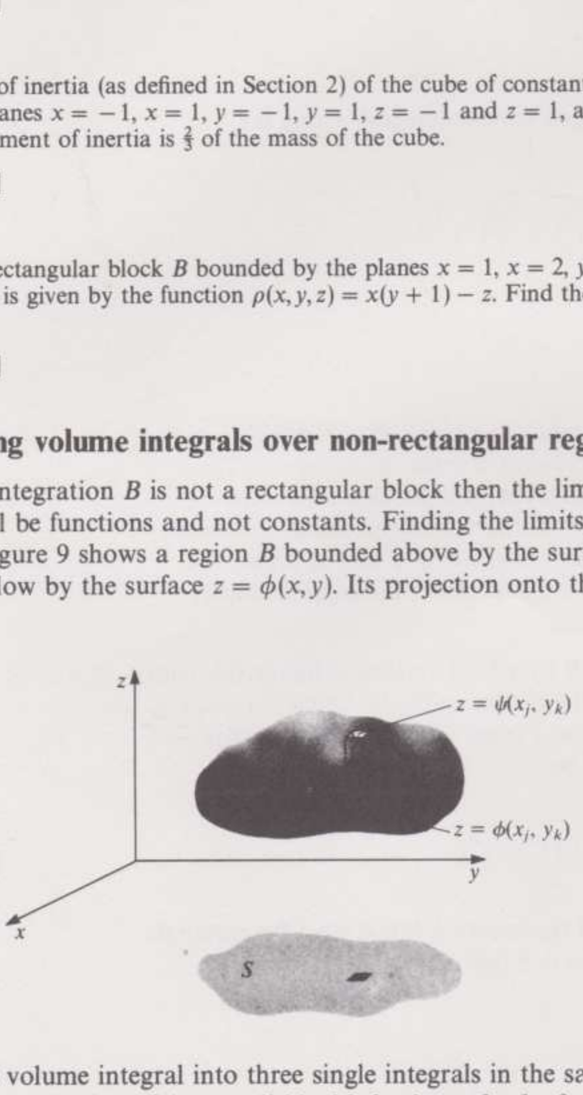


Figure 9

We build up the volume integral into three single integrals in the same way as in the last subsection. Starting with a small block of volume $\delta x_j\,\delta y_k\,\delta z_l$, and summing over the vertical column between $z=\phi(x_j,y_k)$ and $z=\psi(x_j,y_k)$, we have

$$\left[\sum_{l=1}^{r} f(x_j,y_k,z_l)\,\delta z_l\right]\delta x_j\,\delta y_k.$$

The limit of the inner sum as δz_l becomes small is the single integral of f between $z=\phi(x_j,y_k)$ and $z=\psi(x_j,y_k)$, i.e.

$$\int_{z=\phi(x_j,y_k)}^{z=\psi(x_j,y_k)} f(x_j,y_k,z)\,dz.$$

This single integral over z gives a function of x_j and y_k because f and the limits

depend on x_j and y_k. Suppose we denote this function by $g(x_j, y_k)$. Then take all such columns within a slice of width δx_j and sum over the k's to give

$$\left[\sum_{k=1}^{m} g(x_j, y_k)\,\delta y_k\right]\delta x_j.$$

Finally take all such slices and sum over the j's to give a double summation

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{m} g(x_j, y_k)\,\delta y_k\right]\delta x_j.$$

In the limit as δx_j and δy_k become small this double summation is just the surface integral of $g(x, y)$ over the region S of the x,y-plane. The following example illustrates the method.

Example 2

Find the value of the volume integral of the function $f(x, y, z) = y + z$ over the region B within the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ lying in the upper half space $z \geqslant 0$.

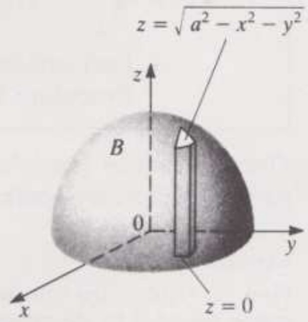


Figure 10

Solution

The volume of integration is shown in Figure 10. The function $\phi(x, y) = 0$, and $\psi(x, y) = \sqrt{a^2 - x^2 - y^2}$. The z integration is taken over the column shown. The bottom of the column (and hence the lower limit) is $z = 0$ and the top of the column (and hence the upper limit) is $z = \sqrt{a^2 - x^2 - y^2}$. The single integral of $f(x, y, z)$ over z is

$$\int_{z=0}^{z=\sqrt{a^2-x^2-y^2}} (y + z)\,dz = \left[yz + \frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-x^2-y^2}}$$
$$= y\sqrt{a^2 - x^2 - y^2} + \frac{(a^2 - x^2 - y^2)}{2}.$$

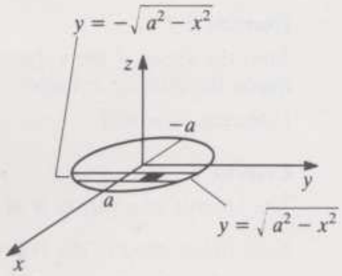


Figure 11

Now we evaluate the surface integral of this function over the projection of B onto the x,y-plane. This projection is just a circle with centre the origin and radius a, as shown in Figure 11. Using Procedure 1.4:

Step 2: The limits on the x integration are

$$x = -a \qquad \text{and} \qquad x = a.$$

Step 3: The limits on the y integration are $y = -\sqrt{a^2 - x^2}$ and $y = \sqrt{a^2 - x^2}$.

Step 4: The surface integral can be written as

$$\int_{x=-a}^{x=a}\left[\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \left(y\sqrt{a^2 - x^2 - y^2} + \frac{1}{2}(a^2 - x^2 - y^2)\right) dy\right] dx.$$

Step 5: Calculating the inner integral first, we get

$$\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \left(y\sqrt{a^2 - x^2 - y^2} + \frac{1}{2}(a^2 - x^2 - y^2)\right) dy$$
$$= \left[\frac{1}{3}(a^2 - x^2 - y^2)^{3/2} + \frac{1}{2}\left((a^2 - x^2)y - \frac{y^3}{3}\right)\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$
$$= (a^2 - x^2)^{3/2} - \frac{1}{3}(a^2 - x^2)^{3/2}$$
$$= \frac{2}{3}(a^2 - x^2)^{3/2}.$$

Step 6: Using this result we calculate the outer integral to give

$$\int_{x=-a}^{x=a} \frac{2}{3}(a^2 - x^2)^{3/2}\,dx = \frac{\pi a^4}{4}.$$

The value of the volume integral is $\pi a^4/4$.

The steps we have used to evaluate a volume integral are contained in the following procedure.

Procedure 3.3: Evaluating volume integrals using Cartesian co-ordinates

1. Draw two diagrams showing (a) the region of integration, B, with the equations of the upper and lower boundaries marked and (b) the projection of this region onto the x,y-plane.
2. Within the region B draw a column, of rectangular cross-section, perpendicular to the x,y-plane and mark clearly the limits of the z integration, $z = \phi(x, y)$ and $z = \psi(x, y)$, say, if not already obvious.
3. Evaluate the single integral of $f(x, y, z)$ over z between $z = \phi(x, y)$ and $z = \psi(x, y)$, keeping x and y constant, i.e. find the function $g(x, y)$ defined by
$$g(x, y) = \int_{z=\phi(x,y)}^{z=\psi(x,y)} f(x, y, z)\, dz.$$
4. Evaluate the surface integral of $g(x, y)$ over the region S using Procedure 1.4.

There are similar procedures in which the 'column' is perpendicular to the x,z- or y,z-planes. Here are some exercises for you to try.

Exercise 4

Find the value of the volume integral of the function $f(x, y, z) = x^2yz$ over the wedge-shaped region bounded by the planes $y + z = 1$, $z = 0$, $y = 0$, $x = 0$ and $x = 1$ (as shown in Figure 12) by going through the steps in Procedure 3.3.

[*Solution on p. 46*]

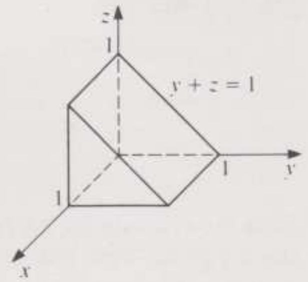


Figure 12

Exercise 5

Find the value of the volume integral of the function $f(x, y, z) = z + 3x - 2$ over the region inside the circular cylinder $x^2 + y^2 = 1$ and lying between the planes $z = 0$ and $z = 1$.

[*Solution on p. 47*]

Exercise 6

The volume of a region B of space can be written in terms of the volume integral as $\int_B 1\, dV$.

Find the volume of the region B inside the surface $z = \frac{1}{9}(x^2 + y^2)$ and below the plane $z = 1$. $\left(\text{You may assume that } \int_{x=-3}^{x=3} (9 - x^2)^{3/2}\, dx = \frac{243}{8}\pi.\right)$

[*Solution on p. 47*]

Summary of Section 3

Procedure 3.3 can be used to evaluate a volume integral using Cartesian co-ordinates.

4 Changing variables in three dimensions

In Section 2 we saw that for some surface integrals where the region of integration was part of a circle it was more convenient to work in polar co-ordinates than in Cartesian co-ordinates. The best choice for a co-ordinate system will depend on the form of the function to be integrated and the shape of the region of integration.

In the television programme we introduced two three-dimensional polar co-ordinate systems—cylindrical polar co-ordinates and spherical polar co-ordinates. We suggested that for regions of integration which have cylindrical or spherical symmetry the use of these co-ordinate systems may simplify the algebra. In this section we will review the definition of cylindrical and spherical polar co-ordinates and use them to evaluate some volume integrals including those introduced in the television programme.

4.1 Cylindrical polar co-ordinates

This co-ordinate system extends the familiar two-dimensional polar co-ordinates (r, θ) to three dimensions by choosing the z-axis of the Cartesian co-ordinate system as the third variable in the following way:

> **Cylindrical polar co-ordinates:** Any point P can be represented by the triple (r, θ, z) where z is the distance of P from the x,y-plane and (r, θ) are the usual polar co-ordinates of the projection of P onto the x,y-plane (see Figure 1). The algebraic relation of cylindrical polar co-ordinates to Cartesian co-ordinates is given by
>
> $$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

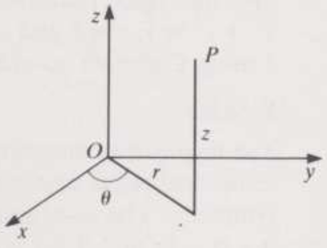


Figure 1

In this definition r, θ and z are allowed to take values in the following ranges:

$$r \geqslant 0, \quad 0 \leqslant \theta < 2\pi, \quad z \text{ can take any real value.}$$

In Cartesian co-ordinates the surfaces x = constant, y = constant and z = constant are planes. They were important when defining the volume element in Section 3. We found there that it was convenient when working with Cartesian co-ordinates to choose as a volume element a small block of sides δx, δy, δz. The faces of this block were area elements drawn in the planes x = constant, y = constant and z = constant.

To express a volume integral in cylindrical polar co-ordinates we must first choose an appropriate shape for the volume element δV. To do this we use a construction similar to that for Cartesian co-ordinates.

The surfaces r = constant are circular cylinders (see Figure 2). The surfaces θ = constant are planes containing the z-axis (see Figure 3). The surfaces z = constant are planes perpendicular to the z-axis (see Figure 4).

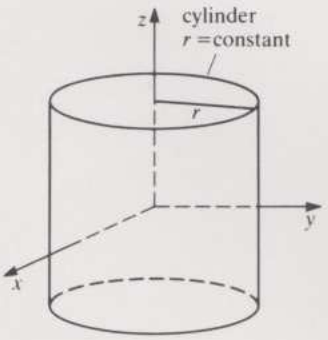


Figure 2

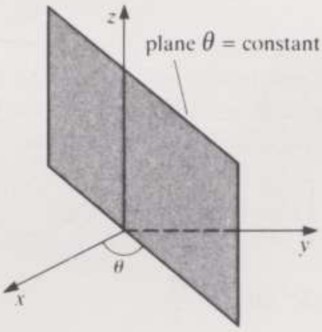


Figure 3

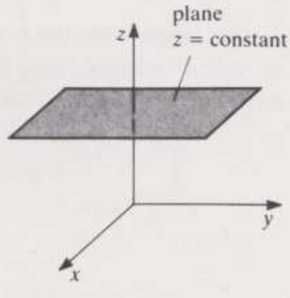


Figure 4

A volume element is shown in Figure 5. Each face of this element is part of a surface of one of the three types described above. The shape of the shaded area is the same as that used in Section 2 as an area element in polar co-ordinates. The area of this element is $r\,\delta\theta\,\delta r$. The height, or depth, of the volume element is δz and so its volume is

$$\boxed{\delta V = r\,\delta r\,\delta\theta\,\delta z.}$$

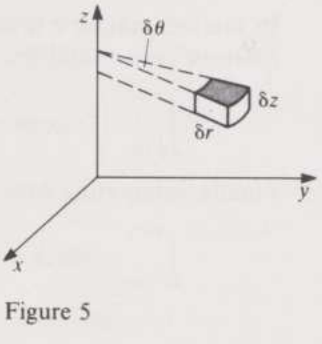


Figure 5

We shall illustrate the method of evaluating volume integrals using cylindrical polar co-ordinates in the following two examples.

Example 1

Find the value of the volume integral of the function

$$f(x, y, z) = z + 3x - 2$$

over the region enclosed within the circular cylinder bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = 1$. (This integral was evaluated in Exercise 5 of Section 3 using Cartesian co-ordinates.)

Solution

The region B of integration is shown in Figure 6. This example is best solved using cylindrical polar co-ordinates because the region of integration has cylindrical symmetry. The co-ordinates r, θ and z take values satisfying $0 \leqslant r \leqslant 1$, $0 \leqslant \theta < 2\pi$ and $0 \leqslant z \leqslant 1$. The function $f(x, y, z)$ becomes

$$f(r\cos\theta, r\sin\theta, z) = z + 3r\cos\theta - 2$$

in cylindrical polar co-ordinates. The volume integral we have to evaluate is

$$\int_B f\,dV = \int_B (z + 3r\cos\theta - 2)r\,dr\,d\theta\,dz.$$

Now to evaluate this we will obtain three single integrals, one over r, one over θ and one over z. These integrals can be performed in different orders depending on how we build up the volume elements to form the region of integration.

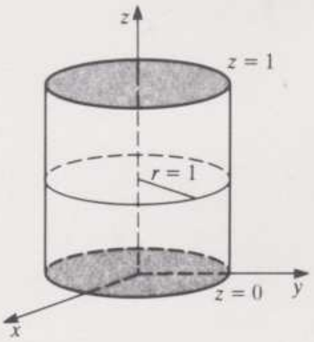


Figure 6

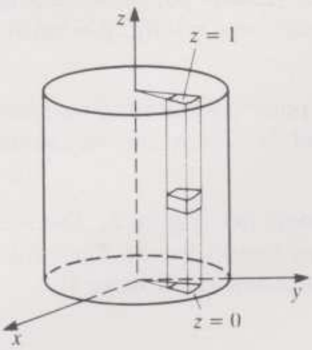


Figure 7

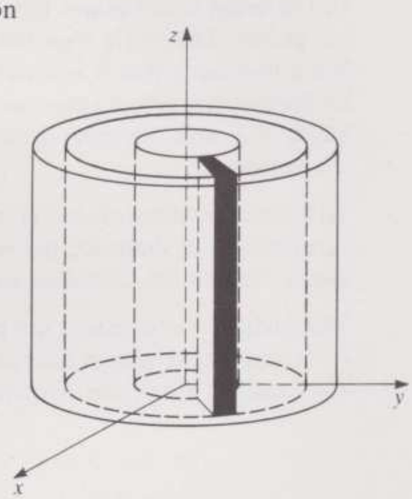


Figure 8

Let us begin with the z integral. This means considering all the volume elements in a vertical column lying between the planes $z = 0$ and $z = 1$ (see Figure 7). In this column r and θ are constant, so we evaluate the single integral

$$\int_{z=0}^{z=1} (z + 3r\cos\theta - 2)r\,dz = \left[r\frac{z^2}{2} + r(3r\cos\theta - 2)z\right]_{z=0}^{z=1}$$

$$= 3r^2\cos\theta - \frac{3r}{2}.$$

Next take the θ integral. θ takes values between $\theta = 0$ and 2π so we have

$$\int_{\theta=0}^{\theta=2\pi} \left(3r^2\cos\theta - \frac{3r}{2}\right) d\theta.$$

In this integration r is constant and the vertical column generates a tube by rotation once round the z-axis (see Figure 8). Now

$$\int_{\theta=0}^{\theta=2\pi} \left(3r^2\cos\theta - \frac{3r}{2}\right) d\theta = \left[3r^2\sin\theta - \frac{3r}{2}\theta\right]_{\theta=0}^{\theta=2\pi} = -3\pi r.$$

Finally, integrating over r between the limits $r = 0$ and 1 we have

$$\int_{r=0}^{r=1} -3\pi r\,dr = \left[-\frac{3\pi r^2}{2}\right]_{r=0}^{r=1} = -\frac{3\pi}{2}.$$

Putting these three single integrals together we can write

$$\int_B f\,dV = \int_{r=0}^{r=1}\left(\int_{\theta=0}^{\theta=2\pi}\left[\int_{z=0}^{z=1}(z + 3r\cos\theta - 2)r\,dz\right]d\theta\right)dr = -\frac{3\pi}{2}.$$

This is the same answer as we had before but the calculation is simpler because here the limits of integration are constants. The limits on the three integrals are all constant because the region of integration is a cylindrical solid.

The method is the same for problems where the region is not a cylinder and the following example illustrates this.

Example 2

Find the value of the volume integral of the function

$$f(x,y,z) = x^2 + y^2$$

over the region bounded by the surfaces $z = x^2 + y^2$, $x = 0$, $y = 0$ and $z = 1$.

Solution

An important part of solving these problems is drawing a diagram showing the region of integration. This helps us to find the limits of integration. For this problem the region is shown in Figure 9.

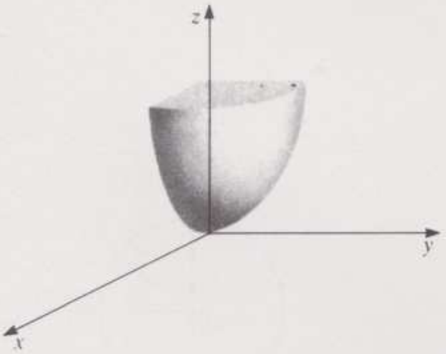


Figure 9

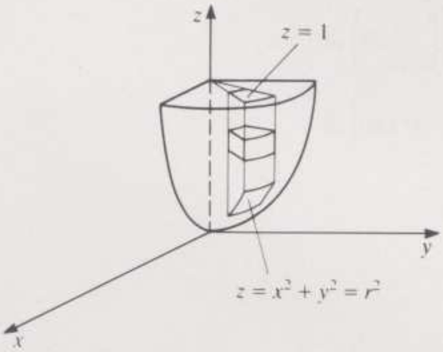


Figure 10

Cylindrical polar co-ordinates are used in this problem because it is easy to describe the region of integration in this system. In cylindrical polar co-ordinates the function to be integrated, $x^2 + y^2$, becomes r^2 and the volume integral becomes

$$\int_B (r^2)r\,dr\,d\theta\,dz.$$

To write this in terms of three single integrals—one over z, one over θ and one over r—we have to find the limits.

Let us start with the z integral. The column, formed by the volume elements, has its lower end at the curved surface $z = x^2 + y^2$, which in cylindrical polar co-ordinates becomes $z = r^2$, and its upper end in the plane $z = 1$ (see Figure 10). So the limits on the z integration are $z = r^2$ and $z = 1$, and the contribution of this column to the volume integral is

$$\left[\int_{z=r^2}^{z=1} r^3\,dz\right]\delta r\,\delta\theta.$$

Next we do the θ integration. The two vertical planes bounding the region are $x = 0$ and $y = 0$. These planes are defined by $\theta = 0$ and $\theta = \pi/2$ in cylindrical polar co-ordinates; so the limits on the θ integration are $\theta = 0$ and $\theta = \pi/2$, and the column generates a quadrant of a tube (see Figure 11). The contribution of this tube to the volume integral is

$$\left(\int_{\theta=0}^{\theta=\pi/2}\left[\int_{z=r^2}^{z=1} r^3\,dz\right]d\theta\right)\delta r.$$

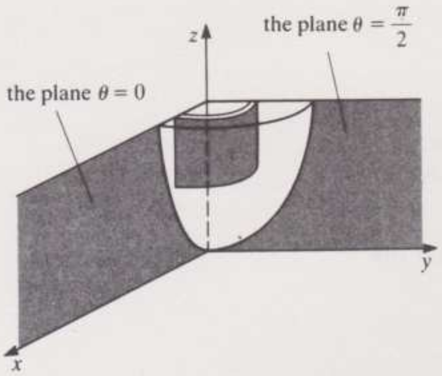


Figure 11

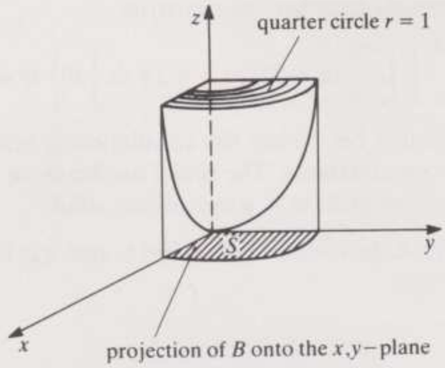


Figure 12

The outer edge of the region corresponds to $r = 1$, so the limits on the r integration are $r = 0$ and $r = 1$, and as r varies from 0 to 1 the quarter tubes fill the whole region (see Figure 12). So we have

$$\begin{aligned}
&\int_{r=0}^{r=1} \left(\int_{\theta=0}^{\theta=\pi/2} \left[\int_{z=r^2}^{z=1} r^3 \, dz \right] d\theta \right) dr \\
&= \int_{r=0}^{r=1} \left(\int_{\theta=0}^{\theta=\pi/2} \left[r^3 z \right]_{z=r^2}^{z=1} d\theta \right) dr \\
&= \int_{r=0}^{r=1} \left(\int_{\theta=0}^{\theta=\pi/2} r^3(1 - r^2) \, d\theta \right) dr \\
&= \int_{r=0}^{r=1} \frac{\pi}{2}(r^3 - r^5) \, dr \\
&= \frac{\pi}{24}.
\end{aligned}$$

In each of these examples the r and θ limits are best visualized by drawing the projection S of the region of integration onto the x,y-plane. Then put on the equations of the lines bounding this area in terms of $r =$ constant and $\theta =$ constant.

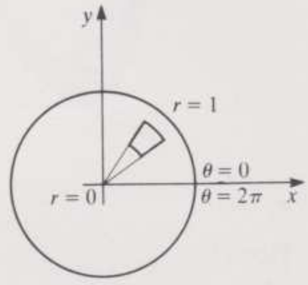


Figure 13

For instance, in Example 1 the projection of the cylinder is a circle of radius 1 (see Figure 13). So that the area elements will cover each point in the circle, θ takes values between 0 and 2π (these are the θ limits) and r takes values between 0 and 1 (these are the r limits).

Figure 14

In Example 2, the projection of the region of integration is a quarter circle. The limits of integration are clearly shown on Figure 14.

We summarize the method as follows:

Procedure 4.1: Evaluating volume integrals using cylindrical polar co-ordinates

1. Draw two diagrams showing (a) the region of integration and (b) the projection of this region onto the x,y-plane.
2. Write the volume integral in terms of r, θ and z using the volume element $\delta V = r\,\delta r\,\delta\theta\,\delta z$.
3. Find the limits for the z, θ and r integrations. Start with the z integral and draw in a column parallel to the z-axis. The limits of the z integration are the values of z where this column intersects the upper and lower surfaces bounding the region of integration. The r and θ limits are found from the projection of the region of integration onto the x,y-plane.
4. Evaluate the three single integrals.

Steps 1 and 3 are perhaps the most difficult part of the procedure. However if you can draw a good diagram of the region of integration then Step 3 becomes relatively easy.

Exercise 1

Find the moment of inertia of a circular cylinder B, of constant density ρ and radius 3 m and height 5 m, about its axis of symmetry.

[*Solution on p. 48*]

Exercise 2

Find the volume integral of the function $f(x, y, z) = z$ over the region B bounded by the surfaces $z = \sqrt{x^2 + y^2}$, $x = 0$, $x - y = 0$ and $z = 4$, as shown in Figure 15.

[*Solution on p. 48*]

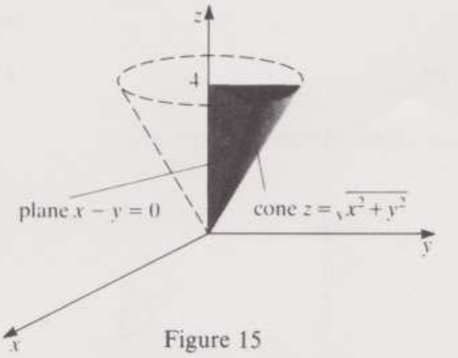


Figure 15

4.2 Spherical polar co-ordinates

This co-ordinate system is based upon the geometry of a sphere and is defined in the following way.

> **Spherical polar co-ordinates**: The position of the point P (see Figure 16) can be given in terms of the distance r of P from the origin and the two angles θ and ϕ. (In Figure 16, Q is the projection of P onto the x,y-plane.) If x, y and z are the Cartesian co-ordinates of P then from Figure 16 we see that
>
> $$x = OQ\cos\phi = r\sin\theta\cos\phi$$
>
> $$y = OQ\sin\phi = r\sin\theta\sin\phi,$$
>
> $$z = r\cos\theta.$$
>
> We require that
>
> $$r \geqslant 0, \qquad 0 \leqslant \phi \leqslant 2\pi \qquad \text{and} \qquad 0 \leqslant \theta \leqslant \pi.$$

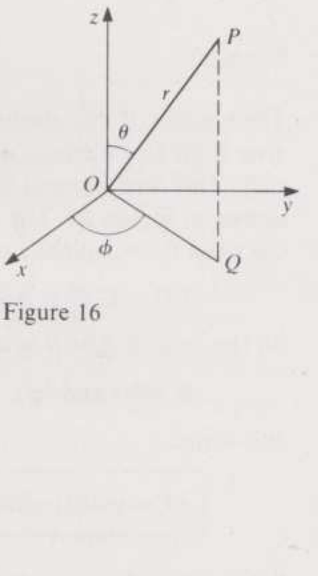


Figure 16

The reason for the name 'spherical polar co-ordinates' is because the equation $r = $ constant represents a sphere.

Exercise 3

(i) In Cartesian co-ordinates (x, y, z) the points inside a sphere of radius a are such that

$$x^2 + y^2 + z^2 \leqslant a^2.$$

How would this sphere be defined using r, θ and ϕ?

(ii) Draw a picture of the solid defined by the set of points (r, θ, ϕ) such that

$$0 \leqslant r \leqslant a, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{4}, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

[*Solution on p. 48*]

To use spherical polar co-ordinates in volume integrals we must find the shape of the volume element and its volume δV. For cylindrical polar co-ordinates we chose the boundary surface of the volume element to lie in the surfaces $r = $ constant, $\theta = $ constant, $z = $ constant.

We apply a similar construction here. The surface r = constant is a sphere (see Figure 17). The surface θ = constant is a cone (see Figure 18). The surface ϕ = constant is a plane containing the z-axis (see Figure 19).

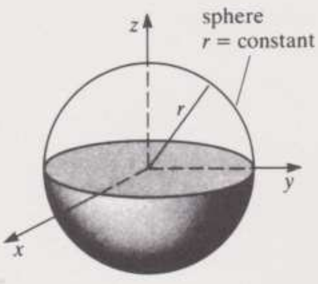


Figure 17

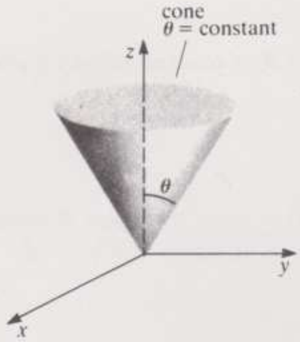


Figure 18

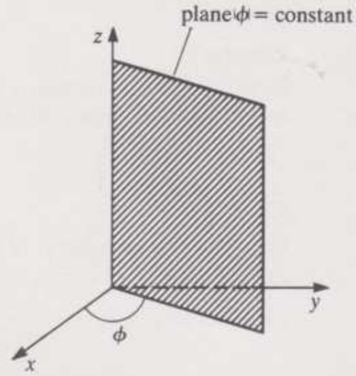


Figure 19

A volume element is shown in Figure 20. Each face of this element is part of a surface of one of the three types described above.

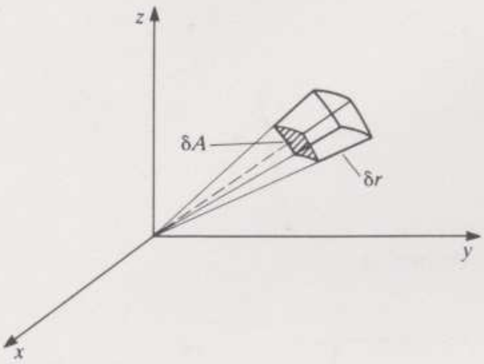


Figure 20

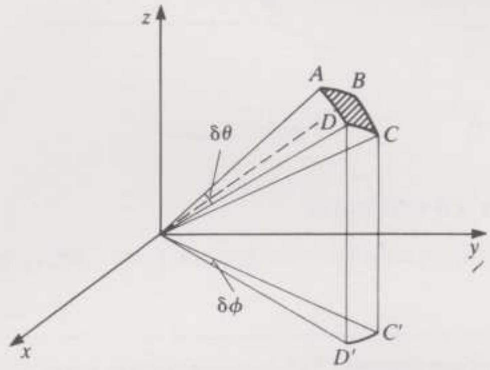


Figure 21

The volume of this element is the product of the shaded area and δr. The shaded area is an area element of a sphere for which r = constant, so that only θ and ϕ vary. This area element is formed by small changes in θ and ϕ, i.e. $\delta\theta$ and $\delta\phi$ as shown in Figure 21. The length of AD is just $r\,\delta\theta$. Now $C'D'$ is the projection of CD onto the x,y-plane and

$$CD = C'D' = (r \sin\theta)\,\delta\phi.$$

So the area of $ABCD$ is approximately equal to

$$(r\,\delta\theta)\,(r \sin\theta\,\delta\phi)$$

and hence

$$\boxed{\delta V = (r\,\delta\theta)\,(r \sin\theta\,\delta\phi)\,\delta r = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi.}$$

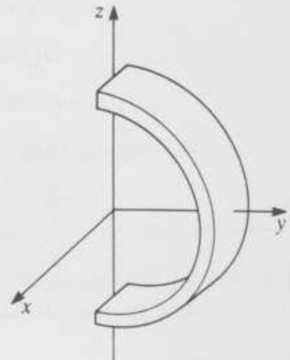


Figure 22

When using this co-ordinate system to evaluate volume integrals, finding the limits can be tricky because of the difficulty of visualizing what is going on.

Before we tackle some problems let us see what is going on when we integrate over the variables r, θ, ϕ when the region of integration is a sphere of radius a. Remember that a volume integral can be approximated by a summation and in the summing process we have to make sure that the volume elements occupy every point within the region of integration.

Now the ordering of the summation can be taken in various ways. If we integrate over θ first from $\theta = 0$ to $\theta = \pi$, holding r and ϕ constant, we obtain a strip bent in the form of an arc of a circle (see Figure 22). If we next integrate over ϕ from $\phi = 0$ to $\phi = 2\pi$, holding r constant, then this strip forms a spherical shell (see Figure 23). And finally integrating over r from $r = 0$ to $r = a$ we form the entire spherical region of integration.

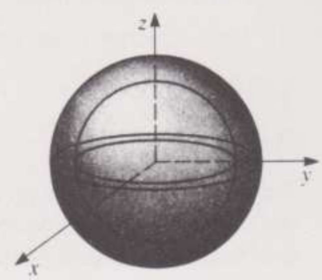


Figure 23

We use spherical polar co-ordinates when the region of integration is easily described in this co-ordinate system. The following example illustrates the method; we shall integrate over a sphere of radius a, and so we can use the method just described.

Example 3

Find the moment of inertia of a solid sphere B of uniform density ρ and radius a about an axis passing through its centre.

Solution

Choose the axis to be the z-axis so that the distance of the point (x, y, z) from the axis is $\sqrt{x^2 + y^2}$ and the moment of inertia of the sphere becomes

$$\int_B \rho(x^2 + y^2)\, dV.$$

In spherical polar co-ordinates

$$x^2 + y^2 = r^2\sin^2\theta$$

and $\quad \delta V = r^2\sin\theta\, \delta r\, \delta\theta\, \delta\phi$

so that the volume integral becomes

$$\int_B \rho(r^2\sin^2\theta)(r^2\sin\theta\, dr\, d\theta\, d\phi).$$

For the θ integral the limits of integration are $\theta = 0$ and $\theta = \pi$. So we have

$$\begin{aligned}\int_{\theta=0}^{\theta=\pi} \rho r^4\sin^3\theta\, d\theta &= \int_{\theta=0}^{\theta=\pi} \rho r^4\sin\theta(1 - \cos^2\theta)\, d\theta \\ &= \left[-\rho r^4\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\right]_{\theta=0}^{\theta=\pi} \\ &= \tfrac{4}{3}r^4\rho.\end{aligned}$$

The limits of the ϕ integration are $\phi = 0$ and $\phi = 2\pi$. So we have

$$\int_{\phi=0}^{\phi=2\pi} \tfrac{4}{3}r^4\rho\, d\phi = \frac{8\pi r^4\rho}{3}.$$

Finally the r integration is between the limits $r = 0$ and $r = a$. Thus

$$\begin{aligned}\int_B \rho(x^2 + y^2)\, dV &= \int_{r=0}^{r=a}\left[\int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi} \rho r^4\sin^3\theta\, d\theta\right) d\phi\right] dr \\ &= \int_{r=0}^{r=a} \frac{8\pi\rho r^4}{3}\, dr \\ &= \frac{8\pi\rho a^5}{15}.\end{aligned}$$

This is the value quoted in Section 2.

The moment of inertia is usually written in terms of the mass of the body rather than its density. For the sphere the mass is $\tfrac{4}{3}\pi\rho a^3$ so that the moment of inertia is $\tfrac{2}{5}Ma^2$.

We summarize the method used in this example in the following procedure:

Procedure 4.2: Evaluating volume integrals using spherical polar co-ordinates

1. Draw a diagram showing the region of integration.
2. Write the volume integral in terms of r, θ, ϕ using the volume element

$$\delta V = r^2\sin\theta\, \delta r\, \delta\theta\, \delta\phi.$$

3. Find the limits for the r, θ and ϕ integrations.
4. Evaluate the three single integrals.

Exercise 4

Find the mass of a solid hemisphere B of constant density ρ and radius a, expressing it as a volume integral in spherical polar co-ordinates.

[*Solution on p. 48*]

Exercise 5

Find the moment of inertia of a spherical shell B of constant density ρ and external radius a and internal radius b about an axis through its centre.

[*Solution on p. 49*]

4.3 A problem for *Unit 30*

Problem statement

Figure 24 represents a solid sphere B, of radius R, and a fixed arbitrary point P outside the sphere. The position vector of the point P relative to the centre O of the sphere is denoted by $\mathbf{r}$ and the position vector relative to O of a point P' in a volume element δV inside the sphere is denoted by $\mathbf{r}'$. The distribution of the material of the sphere is *spherically symmetric* so that the density, $\rho(\mathbf{r}')$, of the volume element δV depends only on its distance from O,

$$\rho(\mathbf{r}') = \rho(|\mathbf{r}'|).$$

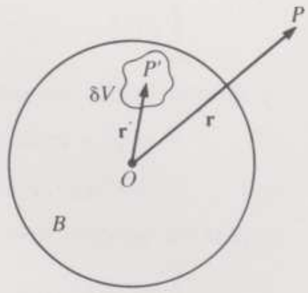


Figure 24

We will show that

$$\int_B \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, dV = \frac{1}{|\mathbf{r}|}\int_B \rho(\mathbf{r}')\, dV.$$

The left-hand volume integral (which I shall call I) is an important one in the theory of gravitation. It expresses the gravitational effect of a spherically symmetric body on a particle placed outside the sphere's surface. The sphere might be a model of the Earth and the outer particle could represent a satellite. You will see in *Unit 30* that for problems involving the motion of a satellite near the Earth, the above result allows us to show that the Earth attracts a small body as if all the mass of the Earth were concentrated at its centre. But that story is for *Unit 30*. Here we will prove the result.

Solution

The problem statement is posed in vector form and implies no particular co-ordinate system. Since the region of integration is a sphere we will use spherical polar co-ordinates (r', θ, ϕ). The calculation is much easier if we choose the axis from which θ is measured to pass through the point P, i.e. to be along OP. This is the $\mathbf{k}$ direction (see Figure 25).

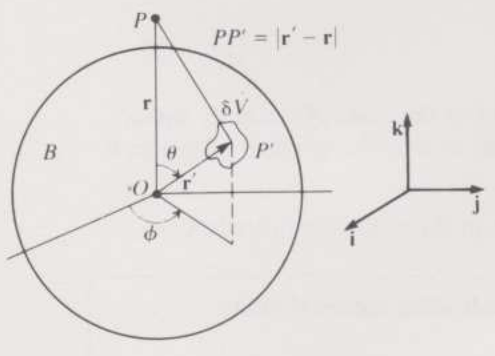


Figure 25

The position vector of P is $\mathbf{r} = r\mathbf{k}$, where $r = |\mathbf{r}|$, and is constant. Similarly the position vector of P' is $\mathbf{r}' = r'\hat{\mathbf{r}}'$, where $r' = |\mathbf{r}'|$.

In spherical polar co-ordinates we know that the position vector of the volume element at P' is

$$\mathbf{r}' = r'\sin\theta\cos\phi\,\mathbf{i} + r'\sin\theta\sin\phi\,\mathbf{j} + r'\cos\theta\,\mathbf{k}$$

so that $|\mathbf{r}' - \mathbf{r}| = (r^2 + r'^2 - 2rr'\cos\theta)^{1/2}$.

The volume integral I becomes

$$I = \int_B \frac{\rho(r')}{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}}\, dV$$

using $\rho(\mathbf{r}') = \rho(|\mathbf{r}'|) = \rho(r')$.

In this volume integral r is constant and $dV = r'^2 \sin\theta\, dr'\, d\theta\, d\phi$. The region of integration is the set of points inside a sphere of radius R, so we can write the volume integral as three single integrals as follows:

$$I = \int_{r'=0}^{r'=R} \left(\int_{\theta=0}^{\theta=\pi} \left[\int_{\phi=0}^{\phi=2\pi} \frac{\rho(r')r'^2 \sin\theta}{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}}\, d\phi \right] d\theta \right) dr'.$$

Integrating over ϕ gives the factor $\int_0^{2\pi} d\phi = 2\pi$, so that

$$I = 2\pi \int_{r'=0}^{r'=R} \left(\int_{\theta=0}^{\theta=\pi} \frac{\rho(r')r'^2 \sin\theta}{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}}\, d\theta \right) dr'.$$

Integrating over θ gives

$$\begin{aligned}
I &= 2\pi \int_{r'=0}^{r'=R} \rho(r')r'^2 \left[\frac{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}}{rr'} \right]_{\theta=0}^{\theta=\pi} dr' \\
&= 2\pi \int_{r'=0}^{r'=R} \frac{\rho(r')r'}{r} \left[(r^2 + r'^2 + 2rr')^{1/2} - (r^2 + r'^2 - 2rr')^{1/2} \right] dr' \\
&= 2\pi \int_{r'=0}^{r'=R} \frac{2r'^2 \rho(r')}{r}\, dr' \\
&= \frac{1}{r} \int_{r'=0}^{r'=R} 4\pi r'^2 \rho(r')\, dr'.
\end{aligned}$$

Consider now the integral

$$J = \int_B \rho(\mathbf{r}')\, dV = \int_B \rho(r')\, dV \qquad \text{(using the spherical symmetry).}$$

In spherical polar co-ordinates we have

$$\begin{aligned}
J &= \int_{r'=0}^{r'=R} \left(\int_{\theta=0}^{\theta=\pi} \left[\int_{\phi=0}^{\phi=2\pi} \rho(r')r'^2 \sin\theta\, d\phi \right] d\theta \right) dr' \\
&= \int_{r'=0}^{r'=R} \left(\int_{\theta=0}^{\theta=\pi} \rho(r')r'^2 \sin\theta\, 2\pi\, d\theta \right) dr' \\
&= \int_{r'=0}^{r'=R} \left(\rho(r')r'^2 2\pi \Big[-\cos\theta \Big]_0^{\pi} \right) dr' \\
&= \int_{r'=0}^{r'=R} 4\pi r'^2 \rho(r')\, dr'.
\end{aligned}$$

Substituting for this integral in the integral for I we have

$$I = \frac{1}{r} J = \frac{1}{|\mathbf{r}|} J$$

and hence we have proved the formula given in the problem statement.

You can perhaps see the relevance of this result if we put $\int_B \rho(\mathbf{r}')\, dV$ equal to the mass, M, say, of the sphere. We then have

$$\int_B \frac{dm}{|\mathbf{r}' - \mathbf{r}|} = \frac{M}{|\mathbf{r}|}$$

where $\delta m = \rho(\mathbf{r}')\,\delta V$ is the mass of a volume element.

Now $|\mathbf{r}' - \mathbf{r}|$ is the distance of the point P from the volume element at P' and $\dfrac{\delta m}{|\mathbf{r}' - \mathbf{r}|}$ is a scalar field proportional to the gravitational potential at P due to an

amount of mass δm at the point P'. The volume integral $\int_B \frac{dm}{|\mathbf{r}' - \mathbf{r}|}$ gives the effect of *all* the mass within the sphere, and this can be written as $\frac{M}{|\mathbf{r}|}$, i.e. as if all the mass of the sphere were concentrated in a particle placed at the centre of the sphere.

You will see in *Unit 30* that this result leads to the law that the gravitational force on a particle outside a sphere of mass M is the same as the force on the particle due to another particle of mass M placed at the centre of the sphere. It took Newton twenty years to prove this by doing summations. With volume integrals it is somewhat quicker!

Summary of Section 4

1. The cylindrical polar co-ordinates (r, θ, z) of a point P in space are related to its Cartesian co-ordinates (x, y, z) by

$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$z = z.$$

That is, (r, θ) are the plane polar co-ordinates of the projection of P onto the x,y-plane.

The volume element in cylindrical polar co-ordinates is

$$\delta V = r\,\delta r\,\delta\theta\,\delta z.$$

This becomes $r\,dr\,d\theta\,dz$ in the limit.

2. To evaluate a volume integral using cylindrical polar co-ordinates use Procedure 4.1, which is similar to the one given for Cartesian co-ordinates; the only difference is that we use plane polar co-ordinates r, θ in place of plane Cartesian co-ordinates x, y throughout.

3. The spherical polar co-ordinates (r, θ, ϕ) of a point P are related to its Cartesian co-ordinates (x, y, z) by

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta.$$

In this case r is the distance from P to the origin O, θ is the angle between the line OP and the positive z-axis, and ϕ is the angle denoted by θ in the case of cylindrical polar co-ordinates.

The volume element in spherical polar co-ordinates is

$$\delta V = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

This becomes $r^2\sin\theta\,dr\,d\theta\,d\phi$ in the limit.

4. To evaluate a volume integral using spherical polar co-ordinates use Procedure 4.2.

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. $\displaystyle\int_{y=1}^{y=3} x^2y^3\,dy = \left[x^2\frac{y^4}{4}\right]_{y=1}^{y=3} = 20x^2.$

In the integration we treat x as a constant.

2.
$$\int_S x^2y^3\,dA = \int_{x=0}^{x=2}\left(\int_{y=1}^{y=3} x^2y^3\,dy\right)dx$$
$$= \int_{x=0}^{x=2}\left(\left[x^2\frac{y^4}{4}\right]_{y=1}^{y=3}\right)dx$$
$$= \int_{x=0}^{x=2}(20x^2)\,dx$$
$$= \left[20\frac{x^3}{3}\right]_{x=0}^{x=2}$$
$$= 160/3.$$

3.
$$\int_S xe^{xy}\,dA = \int_{x=1}^{x=3}\left(\int_{y=2}^{y=3} xe^{xy}\,dy\right)dx$$
$$= \int_{x=1}^{x=3}\left(\left[e^{xy}\right]_{y=2}^{y=3}\right)dx$$
$$= \int_{x=1}^{x=3}(e^{3x}-e^{2x})\,dx$$
$$= \left[\frac{e^{3x}}{3}-\frac{e^{2x}}{2}\right]_{x=1}^{x=3}$$
$$= \frac{e^9}{3}-\frac{e^6}{2}-\frac{e^3}{3}+\frac{e^2}{2}.$$

4. (i)
$$\int_S x^2y^3\,dA = \int_{y=1}^{y=3}\left(\int_{x=0}^{x=2} x^2y^3\,dx\right)dy$$
$$= \int_{y=1}^{y=3}\left(\left[y^3\frac{x^3}{3}\right]_{x=0}^{x=2}\right)dy$$
$$= \int_{y=1}^{y=3}\left(\frac{8}{3}y^3\right)dy$$
$$= \frac{8}{3}\left[\frac{y^4}{4}\right]_{y=1}^{y=3}$$
$$= 160/3 \qquad \text{(as for Exercise 2).}$$

(ii) $\displaystyle\int_S xe^{xy}\,dA = \int_{y=2}^{y=3}\left(\int_{x=1}^{x=3} xe^{xy}\,dx\right)dy$

Integrating over x first, we integrate by parts holding y constant and get

$$\left[x\frac{e^{xy}}{y}\right]_{x=1}^{x=3} - \int_{x=1}^{x=3}\frac{e^{xy}}{y}\,dx = \left(3\frac{e^{3y}}{y}-\frac{e^y}{y}\right)-\left[\frac{e^{xy}}{y^2}\right]_{x=1}^{x=3}$$
$$= 3\frac{e^{3y}}{y}-\frac{e^y}{y}-\frac{e^{3y}}{y^2}+\frac{e^y}{y^2}.$$

Now we integrate over y to give

$$\int_S xe^{xy}\,dA = \int_{y=2}^{y=3}\left(3\frac{e^{3y}}{y}-\frac{e^y}{y}-\frac{e^{3y}}{y^2}+\frac{e^y}{y^2}\right)dy$$
$$= \int_{y=2}^{y=3}\left(\frac{3e^{3y}-e^y}{y}\right)dy - \int_{y=2}^{y=3}\left(\frac{e^{3y}-e^y}{y^2}\right)dy.$$

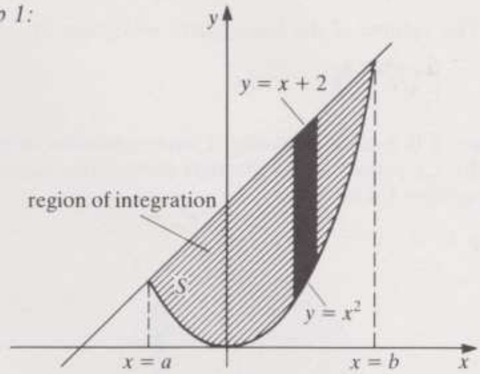
Neither of these two integrals can be evaluated separately, but integrating the second integral by parts and subtracting the answer from the first gives the same answer as in Exercise 3, i.e.

$$\left[\frac{-(e^{3y}-e^y)}{y}\right]_{y=2}^{y=3} - \int_{y=2}^{y=3}(3e^{3y}-e^y)\left(-\frac{1}{y}\right)dy$$
$$= -\frac{e^9}{3}+\frac{e^3}{3}+\frac{e^6}{2}-\frac{e^2}{2}+\int_{y=2}^{y=3}\frac{(3e^{3y}-e^y)}{y}\,dy.$$

Now subtracting this result from the first integral we have

$$\int_S xe^{xy}\,dA = \frac{e^9}{3}-\frac{e^3}{3}-\frac{e^6}{2}+\frac{e^2}{2}.$$

5. *Step 1:*



Step 2: The limits of the x integration are $x = a$ and $x = b$, and these values of x are solutions to the simultaneous equations

$$y = x^2,$$
$$y = x + 2.$$

We have to solve $x^2 - x - 2 = 0$, giving $x = 2$ and $x = -1$, so that $a = -1$ and $b = 2$.

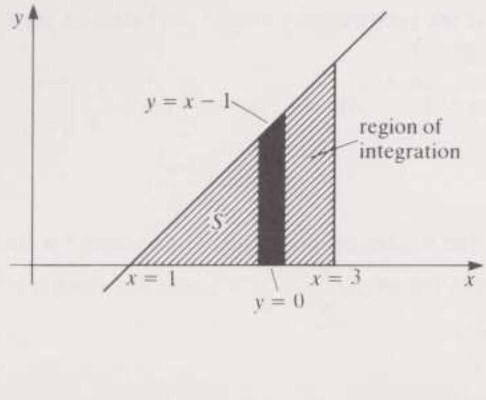
Step 3: To find the y limits we draw in a vertical strip at some general position in the region as shown in the figure above. The lower limit of this strip is then $y = x^2$ and the upper limit is $y = x + 2$.

Step 4: $\displaystyle\int_S y\,dA = \int_{x=-1}^{x=2}\left(\int_{y=x^2}^{y=x+2} y\,dy\right)dx.$

Step 5: $\displaystyle\int_{y=x^2}^{y=x+2} y\,dy = \left[\frac{y^2}{2}\right]_{y=x^2}^{y=x+2} = \frac{(x+2)^2}{2}-\frac{x^4}{2}.$

Step 6:
$$\int_{x=-1}^{x=2}\left(\frac{1}{2}(x+2)^2-\frac{1}{2}x^4\right)dx$$
$$= \left[\frac{1}{6}(x+2)^3-\frac{1}{10}x^5\right]_{x=-1}^{x=2}$$
$$= \frac{36}{5}.$$

6. *Step 1:*



Step 2: The x limits are $x = 1$ and $x = 3$.

Step 3: The y limits are $y = 0$ and $y = x - 1$.

Step 4: $\displaystyle\int_S (x - y)\,dA = \int_{x=1}^{x=3}\left[\int_{y=0}^{y=x-1} (x - y)\,dy\right]dx$

Step 5: $\displaystyle\int_{y=0}^{y=x-1} (x - y)\,dy = \left[xy - \frac{y^2}{2}\right]_{y=0}^{y=x-1}$

$$= x(x-1) - \frac{(x-1)^2}{2}.$$

$$= \frac{x^2}{2} - \frac{1}{2}.$$

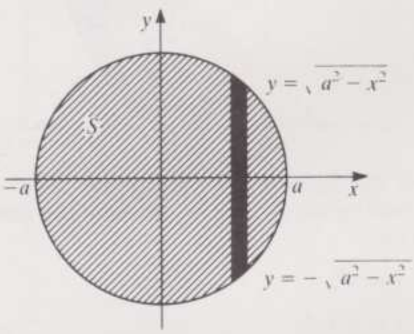
Step 6: $\displaystyle\int_{x=1}^{x=3}\left(\frac{x^2}{2} - \frac{1}{2}\right)dx = \left[\frac{x^3}{6} - \frac{x}{2}\right]_{x=1}^{x=3} = \frac{10}{3}.$

7. The volume of the hemisphere was given by

$$\int_S \sqrt{a^2 - x^2 - y^2}\,dA$$

where S is a circle of radius a, and centre the origin, drawn in the x,y-plane. Going through each of the steps in Procedure 1.4 we have

Step 1:



Step 2: The x limits are $x = -a$ and $x = +a$.

Step 3: The y limits are $y = -\sqrt{a^2 - x^2}$ and $y = +\sqrt{a^2 - x^2}$.

Step 4: $\displaystyle\int_S \sqrt{a^2 - x^2 - y^2}\,dA$

$$= \int_{x=-a}^{x=a}\left(\int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy\right)dx.$$

Step 5: $\displaystyle\int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy$

$$= \frac{a^2 - x^2}{2}\left[\arcsin\left(\frac{y}{\sqrt{a^2 - x^2}}\right) + \frac{y}{\sqrt{a^2 - x^2}}\left(1 - \frac{y^2}{a^2 - x^2}\right)^{1/2}\right]_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}}$$

$$= \frac{(a^2 - x^2)}{2}\pi.$$

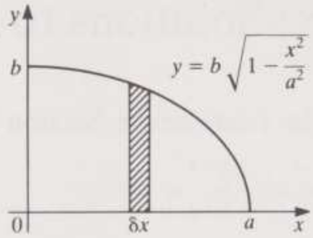
(Use the substitution $y = \sqrt{a^2 - x^2}\sin\theta$, or use a table of integrals.)

Step 6: $\displaystyle\int_{x=-a}^{x=a} \frac{\pi}{2}(a^2 - x^2)\,dx = \frac{\pi}{2}\left[a^2x - \frac{x^3}{3}\right]_{x=-a}^{x=a}$

$$= \frac{2\pi a^3}{3}.$$

8. The double integral is just $\int_S dA$ where S is the area within the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and for which $x \geqslant 0$ and $y \geqslant 0$.

Step 1:



Step 2: The x limits are $x = 0$ and $x = a$.

Step 3: The y limits are $y = 0$ and $y = b\sqrt{1 - \dfrac{x^2}{a^2}}$.

Step 4: $\displaystyle\int_S dA = \int_{x=0}^{x=a}\left(\int_{y=0}^{y=b\sqrt{1-(x^2/a^2)}} dy\right)dx.$

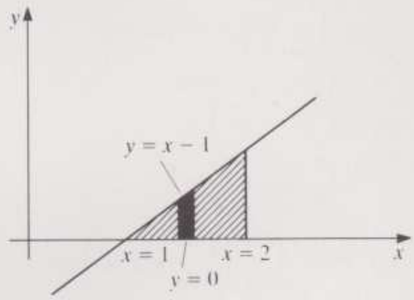
Step 5: $\displaystyle\int_{y=0}^{y=b\sqrt{1-(x^2/a^2)}} dy = \Big[y\Big]_{y=0}^{y=b\sqrt{1-(x^2/a^2)}} = b\sqrt{1 - \frac{x^2}{a^2}}.$

Step 6: $\displaystyle\int_{x=0}^{x=a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx$

$$= \left[\frac{ab}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right)\right]_{x=0}^{x=a} = \frac{\pi ab}{4}.$$

9. (i) If we integrate over y first then we *fix* x and draw in a *vertical* strip as shown.

Step 1:



Step 2: The limits of the x integration are $x = 1$ and $x = 2$.

Step 3: The limits of the y integration are $y = 0$ and $y = x - 1$.

Step 4: $\displaystyle\int_S (x^2 + y^2)\,dA = \int_{x=1}^{x=2}\left[\int_{y=0}^{y=x-1} (x^2 + y^2)\,dy\right]dx.$

Step 5: $\displaystyle\int_{y=0}^{y=x-1} (x^2 + y^2)\,dy = \left[x^2y + \frac{y^3}{3}\right]_{y=0}^{y=x-1}$

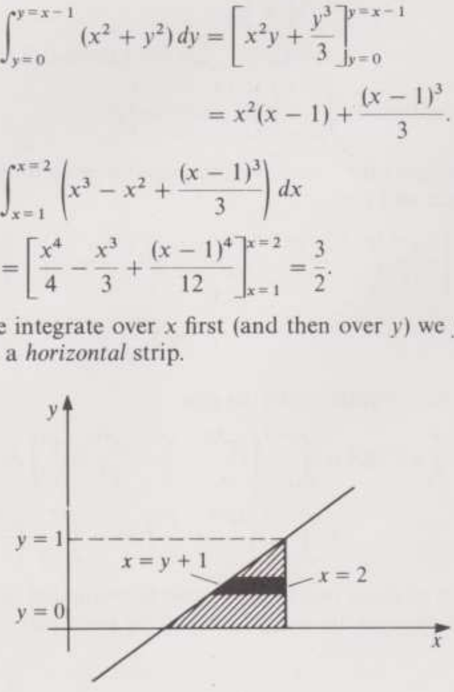
$$= x^2(x-1) + \frac{(x-1)^3}{3}.$$

Step 6: $\displaystyle\int_{x=1}^{x=2}\left(x^3 - x^2 + \frac{(x-1)^3}{3}\right)dx$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} + \frac{(x-1)^4}{12}\right]_{x=1}^{x=2} = \frac{3}{2}.$$

(ii) If we integrate over x first (and then over y) we *fix* y and draw in a *horizontal* strip.

Step 1:



Step 2: The y limits are $y = 0$ and $y = 1$.

Step 3: The x limits are $x = y + 1$ and $x = 2$.

Step 4: $\int_S (x^2 + y^2)\,dA = \int_{y=0}^{y=1} \left[\int_{x=y+1}^{x=2} (x^2 + y^2)\,dx \right] dy$

Step 5: $\int_{x=y+1}^{x=2} (x^2 + y^2)\,dx = \left[\frac{x^3}{3} + xy^2 \right]_{x=y+1}^{x=2}$

$$= \frac{8}{3} + 2y^2 - \frac{(y+1)^3}{3} - (y+1)y^2.$$

Step 6: $\int_{y=0}^{y=1} \left(\frac{8}{3} + 2y^2 - \frac{(y+1)^3}{3} - y^2(1+y) \right) dy$

$$= \left[\frac{8}{3}y + \frac{2y^3}{3} - \frac{(y+1)^4}{12} - \frac{y^3}{3} - \frac{y^4}{4} \right]_{y=0}^{y=1}$$

$$= \frac{3}{2}.$$

Solutions to the exercises in Section 2

1. **(i)** The volume of a sphere of radius a is

$$2 \times \frac{2}{3}\pi a^3 = \frac{4}{3}\pi a^3;$$

if the density is constant then the mass of the sphere is

$$\frac{4}{3}\rho\pi a^3.$$

(ii) The mass of the hollow sphere, M, say, can be calculated by subtracting the mass of a solid sphere of radius b from the mass of a solid sphere of radius a, i.e.

$$M = \frac{4}{3}\pi\rho a^3 - \frac{4}{3}\pi\rho b^3 = \frac{4}{3}\pi\rho(a^3 - b^3).$$

2. **(i)** $I_1 = \frac{8\pi}{15}\rho(a^5 - b^5), \qquad M = \frac{4}{3}\pi\rho(a^3 - b^3);$

hence $\quad \frac{I_1}{M} = \frac{2(a^5 - b^5)}{5(a^3 - b^3)} = \frac{2}{5}a^2 \frac{(1 - (b/a)^5)}{(1 - (b/a)^3)}$

and $\quad \frac{I_2}{M} = \frac{2}{5}a^2$

(either by using the formula in the table or by putting $b = 0$ in $\frac{I_1}{M}$).

(ii) $\frac{I_1}{M} = \frac{I_2}{M}\left[\frac{1 - (b/a)^5}{1 - (b/a)^3}\right]$ and since $0 < b/a < 1$

$$1 - \left(\frac{b}{a}\right)^5 > 1 - \left(\frac{b}{a}\right)^3 \quad \text{and}$$

$$\frac{I_1}{M} > \frac{I_2}{M},$$

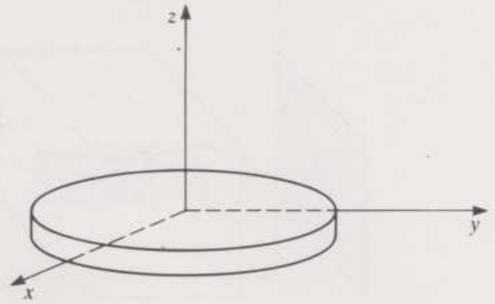
i.e. the moment of inertia of the hollow sphere is greater than that of the solid sphere.

(iii) The faster sphere is the solid sphere.

3. The moment of inertia of the disc is given by

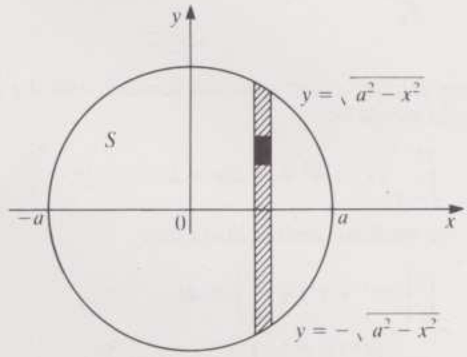
$$\int_S \rho h(x^2 + y^2)\,dA,$$

taking a volume element as $h\,dA$ for this disc.



This is a surface integral so we can use Procedure 1.4.

Step 1: The figure below shows the region of integration, S.



Step 2: The x limits are $x = -a$ and $x = +a$.

Step 3: The y limits are $y = -\sqrt{a^2 - x^2}$ and $y = +\sqrt{a^2 - x^2}$.

Step 4: $\int_S \rho h(x^2 + y^2)\,dA$

$$= \rho h \int_{x=-a}^{x=a} \left[\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} (x^2 + y^2)\,dy \right] dx.$$

Step 5: $\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} (x^2 + y^2)\,dy$

$$= 2x^2\sqrt{a^2 - x^2} + \frac{2}{3}(a^2 - x^2)^{3/2}.$$

Step 6: $\int_{x=-a}^{x=a} \left[2x^2(a^2 - x^2)^{1/2} + \frac{2}{3}(a^2 - x^2)^{3/2} \right] dx$

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} \left(2a^2\sin^2\theta\, a\cos\theta + \frac{2}{3}a^3\cos^3\theta \right) a\cos\theta\,d\theta$$

$$= 2a^4 \int_{\theta=-\pi/2}^{\theta=\pi/2} \left(\frac{\sin^2 2\theta}{4} + \frac{1}{3}\cos^4\theta \right) d\theta$$

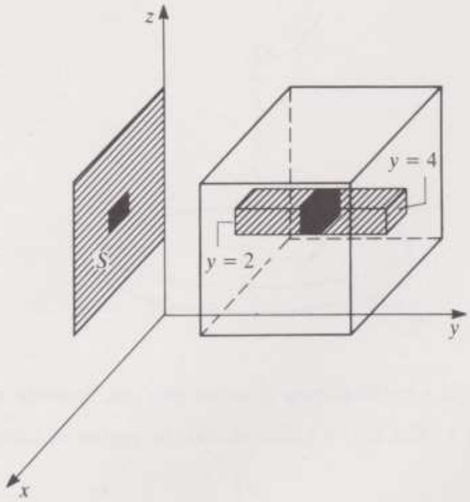
$$= 2a^4 \int_{\theta=-\pi/2}^{\theta=\pi/2} \left[\left(\frac{1 - \cos 4\theta}{8} \right) + \frac{1}{3}\left(\frac{1 + \cos 2\theta}{2} - \left(\frac{1 - \cos 4\theta}{8} \right) \right) \right] d\theta$$

$$= 2a^4 \left(\frac{\pi}{4} \right).$$

So we have $\int_S \rho h(x^2 + y^2)\,dA = \frac{\pi a^4}{2}\rho h.$

Solutions to the exercises in Section 3

1. We could order the volume elements into a column which is parallel to the y,z-plane as shown in the figure below.



In this column x and z remain constant and the first single integral would be

$$\int_{y=2}^{y=4} (x^2 + y^2 + z^2)\,dy = 2(x^2 + z^2) + \frac{56}{3},$$

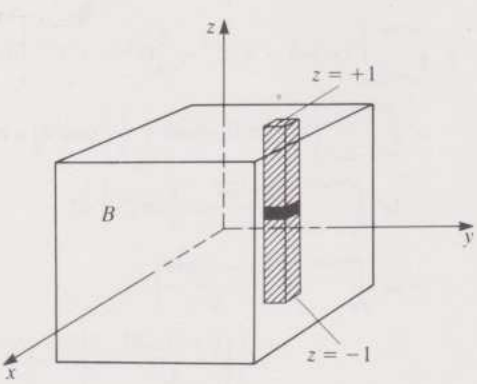
and the resulting surface integral is

$$\int_S \left(2(x^2 + z^2) + \frac{56}{3}\right) dx\,dz$$
$$= \int_{z=1}^{z=3} \left[\int_{x=0}^{x=1} \left(2(x^2 + z^2) + \frac{56}{3}\right) dx \right] dz$$
$$= \int_{z=1}^{z=3} \left(2z^2 + \frac{58}{3}\right) dz$$
$$= 56, \quad \text{as before.}$$

2. The moment of inertia of the cube about the x-axis is given by the volume integral

$$\int_B \rho(y^2 + z^2)\,dV.$$

Forming the volume elements into a column the limits on the z integral are $z = -1$ and $z = +1$.



We have $\int_{z=-1}^{z=1} \rho(y^2 + z^2)\,dz$ as the z integral. The x and y integrals are taken over a square in the x,y-plane. We have

$$\int_{x=-1}^{x=1} \left[\int_{y=-1}^{y=1} \left(\int_{z=-1}^{z=1} \rho(y^2 + z^2)\,dz \right) dy \right] dx$$
$$= \int_{x=-1}^{x=1} \left[\int_{y=-1}^{y=1} \rho \left(2y^2 + \frac{2}{3}\right) dy \right] dx$$
$$= \int_{x=-1}^{x=1} \rho \left(\frac{4}{3} + \frac{4}{3}\right) dx$$
$$= \frac{16}{3}\rho.$$

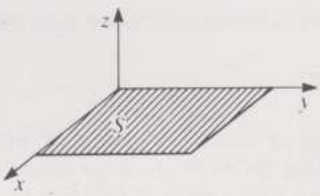
In terms of the mass of the cube, $M = 8\rho$, so we have the moment of inertia about the x-axis equal to $\frac{2}{3}M$.

3. The mass of the block is $\int_B \rho\,dV$. Now

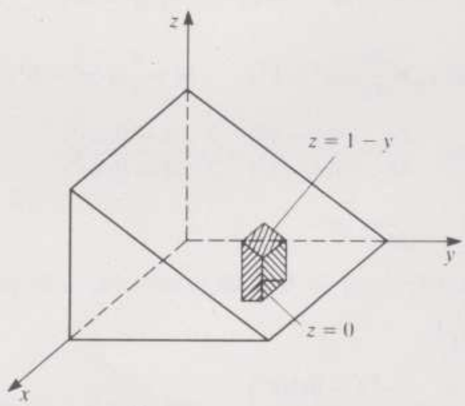
$$\int_B (x(y + 1) - z)\,dV$$
$$= \int_{x=1}^{x=2} \left[\int_{y=0}^{y=3} \left(\int_{z=-1}^{z=0} (x(y + 1) - z)\,dz \right) dy \right] dx$$
$$= \int_{x=1}^{x=2} \left[\int_{y=0}^{y=3} \left(x(y + 1) + \frac{1}{2}\right) dy \right] dx$$
$$= \int_{x=1}^{x=2} \left(3\left(x + \frac{1}{2}\right) + \frac{9}{2}x\right) dx$$
$$= \frac{51}{4}.$$

4. *Step 1:* The region is shown in Figure 12 in the main text. The projection onto the x,y-plane is shown below: it is the square defined by the set of points (x, y) such that

$$0 \leqslant x \leqslant 1, \qquad 0 \leqslant y \leqslant 1.$$



Step 2:

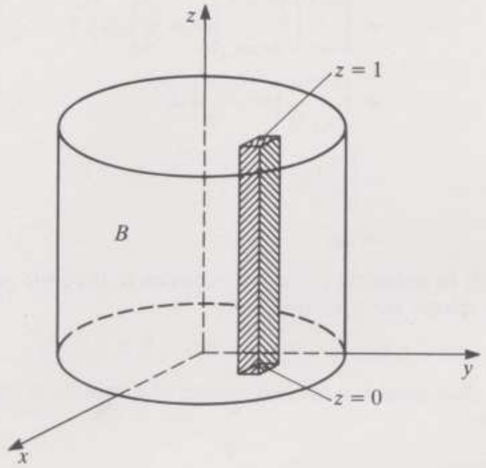


Step 3: $$g(x, y) = \int_{z=0}^{z=1-y} x^2yz\,dz$$
$$= \left[x^2y\frac{z^2}{2}\right]_{z=0}^{z=1-y}$$
$$= \frac{x^2y(1 - y)^2}{2}.$$

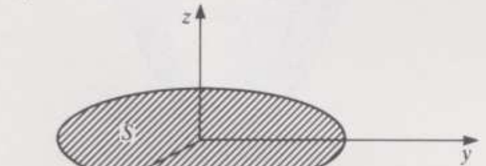
Step 4: Now following Procedure 1.4 for evaluating the double integral of $g(x, y)$ over S,

$$\begin{aligned}\int_S g(x,y)\,dA &= \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1}\frac{x^2y(1-y)^2}{2}\,dy\right]dx\\ &= \int_{x=0}^{x=1}\left[\frac{x^2}{2}\left(\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right)\right]_{y=0}^{y=1}dx\\ &= \int_{x=0}^{x=1}\frac{x^2}{24}\,dx\\ &= 1/72.\end{aligned}$$

5. *Step 1:* The region B is illustrated below:



The projection of B onto the x,y-plane is the circle $x^2 + y^2 \leqslant 1$.

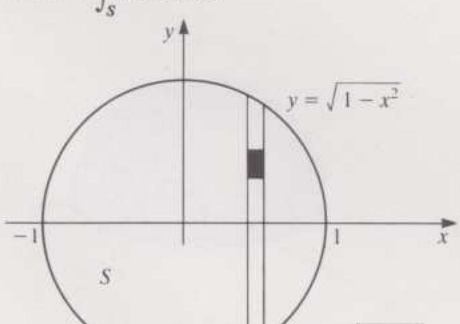


Step 2: The column is shown in the first figure above. The z limits are $z = 0$ and $z = 1$.

Step 3: $\displaystyle\int_{z=0}^{z=1}(z+3x-2)\,dz = \left[\frac{z^2}{2}+(3x-2)z\right]_{z=0}^{z=1}$

$$= \frac{1}{2}+3x-2 = 3x-\frac{3}{2} = g(x,y)$$

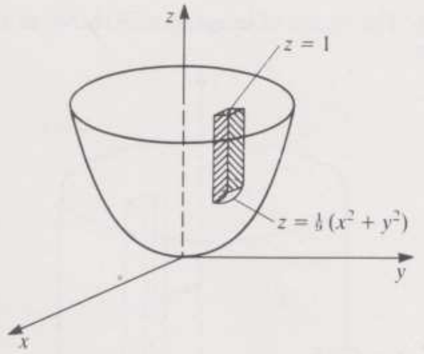
Step 4: Evaluate $\displaystyle\int_S g(x,y)\,dA$.



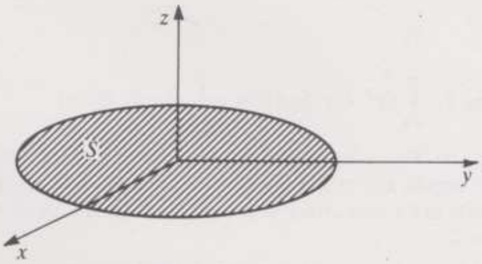
$$\begin{aligned}\int_S g(x,y)\,dA &= \int_{x=-1}^{x=1}\left[\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\left(3x-\frac{3}{2}\right)dy\right]dx\\ &= \int_{x=-1}^{x=1}\left[\left(3x-\frac{3}{2}\right)2\sqrt{1-x^2}\right]dx\\ &= -\frac{3\pi}{2}.\end{aligned}$$

6. The volume is the integral $\displaystyle\int_B dV$.

Step 1: The region B is illustrated below:



The projection of B onto the x,y-plane is the circle $x^2 + y^2 \leqslant 9$.

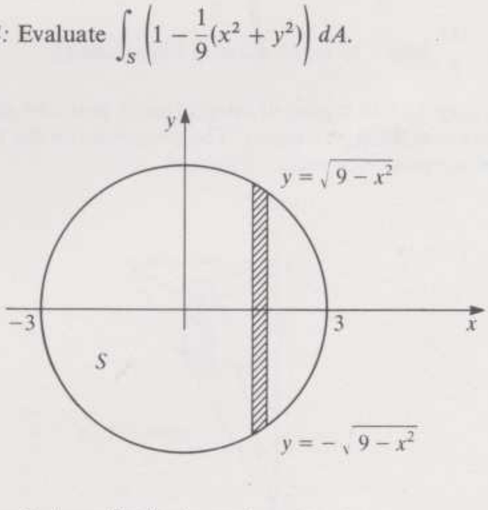


Step 2: A vertical column is shown in the first figure above. The z limits are $z = \dfrac{1}{9}(x^2+y^2)$ and $z = 1$.

Step 3: $f(x,y,z) = 1$ in this exercise. The z integral is

$$\int_{z=(x^2+y^2)/9}^{z=1} dz = 1-\frac{1}{9}(x^2+y^2) = g(x,y).$$

Step 4: Evaluate $\displaystyle\int_S\left(1-\frac{1}{9}(x^2+y^2)\right)dA$.



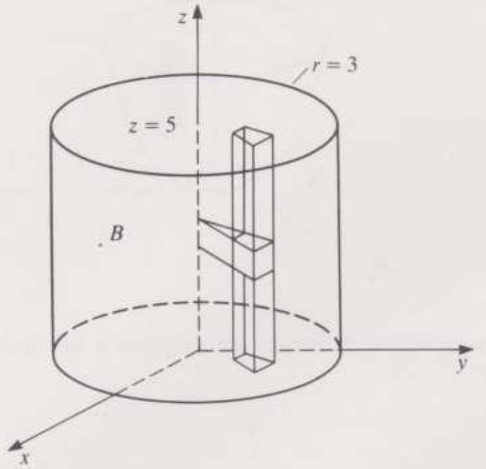
$$\begin{aligned}&\int_{x=-3}^{x=3}\left(\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}\left(1-\frac{x^2}{9}-\frac{y^2}{9}\right)dy\right)dx\\ &= \int_{x=-3}^{x=3}\left[\left(1-\frac{x^2}{9}\right)2\sqrt{9-x^2}-\frac{2}{27}(9-x^2)^{3/2}\right]dx\\ &= \int_{x=-3}^{x=3}\frac{4}{27}(9-x^2)^{3/2}\,dx\\ &= \frac{9\pi}{2}.\end{aligned}$$

Solutions to the exercises in Section 4

1. The moment of inertia of the cylinder about the z-axis is

$$\int_B (x^2 + y^2)\rho\, dV.$$

Step 1: The region of integration is shown in the diagram below.



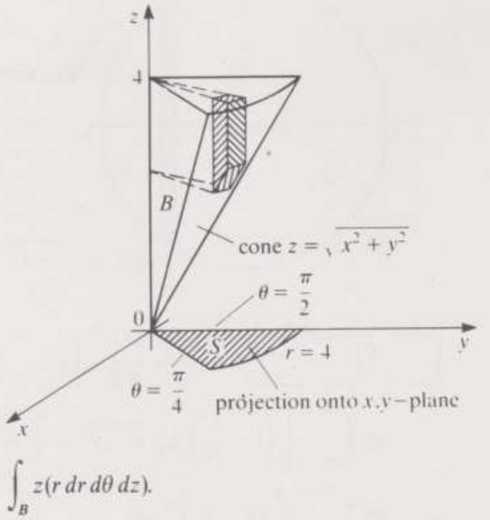
Step 2: $\displaystyle\int_B (x^2 + y^2)\rho\, dV = \rho \int_B r^2 (r\, dr\, d\theta\, dz)$

Step 3: The z limits are $z = 0$ and $z = 5$. The projection of the region on to the x,y-plane is a circle of radius 3. The r limits are $r = 0$ and $r = 3$ and the θ limits are $\theta = 0$ and $\theta = 2\pi$.

$$\begin{aligned}
\textit{Step 4: } \rho \int_B r^3\, dr\, d\theta\, dz &= \rho \int_{r=0}^{r=3} \left(\int_{\theta=0}^{\theta=2\pi} \left(\int_{z=0}^{z=5} r^3\, dz \right) d\theta \right) dr \\
&= \rho \int_{r=0}^{r=3} \left(\int_{\theta=0}^{\theta=2\pi} 5r^3\, d\theta \right) dr \\
&= \rho \int_{r=0}^{r=3} 10\pi r^3\, dr \\
&= \rho \frac{10\pi 3^4}{4} = \frac{405\pi\rho}{2}
\end{aligned}$$

$\left(= \dfrac{9M}{2}\right.$ where M is the mass of the cylinder).

2. *Step 1:* The region of integration is part of a cone as shown in the figure below. The projection of the region onto the x,y-plane is a sector of a circle.



Step 2: $\displaystyle\int_B z (r\, dr\, d\theta\, dz).$

Step 3: The z limits are $z = r$ and $z = 4$.

The r limits are $r = 0$ and $r = 4$.

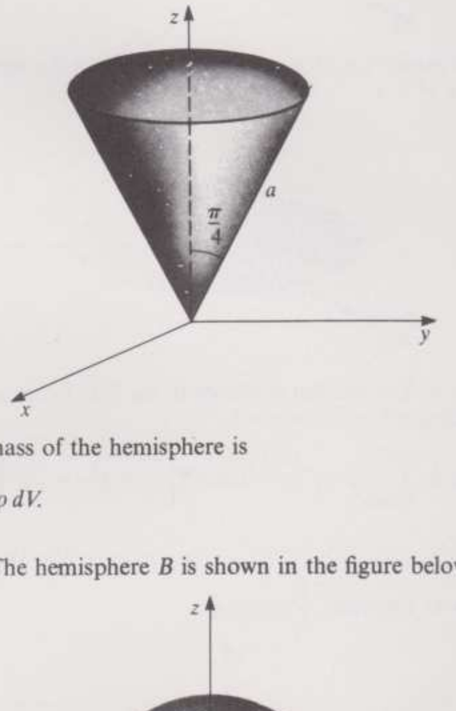
The θ limits are $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{2}$.

$$\begin{aligned}
\textit{Step 4: } & \int_{r=0}^{r=4} \left(\int_{\theta=\pi/4}^{\theta=\pi/2} \left(\int_{z=r}^{z=4} zr\, dz \right) d\theta \right) dr \\
&= \int_{r=0}^{r=4} \left(\int_{\theta=\pi/4}^{\theta=\pi/2} \left[\frac{z^2 r}{2} \right]_{z=r}^{z=4} d\theta \right) dr \\
&= \int_{r=0}^{r=4} \left(\int_{\theta=\pi/4}^{\theta=\pi/2} \left(8r - \frac{r^3}{2} \right) d\theta \right) dr \\
&= \int_{r=0}^{r=4} \frac{\pi}{4} \left(8r - \frac{r^3}{2} \right) dr \\
&= \left[\frac{\pi}{4} \left(4r^2 - \frac{r^4}{8} \right) \right]_{r=0}^{r=4} \\
&= 8\pi.
\end{aligned}$$

3. (i) In spherical polar co-ordinates (r, θ, ϕ) the points inside the sphere are such that

$$0 \leqslant r \leqslant a, \qquad 0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

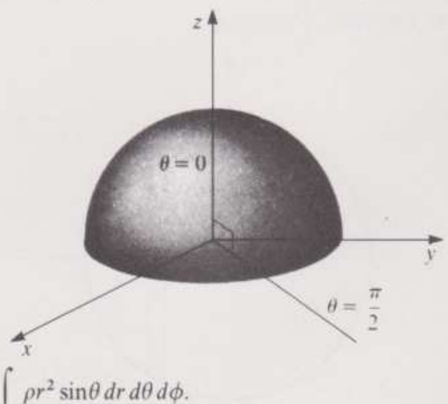
(ii) The solid is a conical section of a sphere, as shown below.



4. The mass of the hemisphere is

$$\int_B \rho\, dV.$$

Step 1: The hemisphere B is shown in the figure below.



Step 2: $\displaystyle\int_B \rho r^2 \sin\theta\, dr\, d\theta\, d\phi.$

Step 3: The θ limits are $\theta = 0$ and $\theta = \pi/2$.

The ϕ limits are $\phi = 0$ and $\phi = 2\pi$.

The r limits are $r = 0$ and $r = a$.

Step 4: $\displaystyle\int_{r=0}^{r=a}\left(\int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi/2}\rho r^2\sin\theta\,d\theta\right)d\phi\right)dr$

$$= \int_{r=0}^{r=a}\left(\int_{\phi=0}^{\phi=2\pi}\rho r^2\,d\phi\right)dr$$

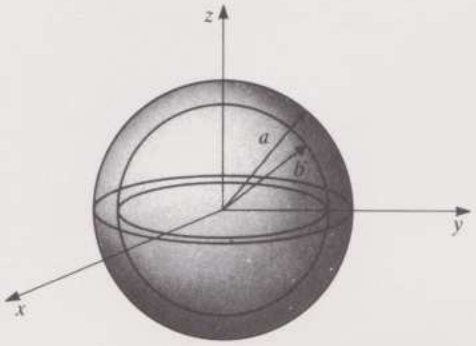
$$= \int_{r=0}^{r=a}2\pi\rho r^2\,dr$$

$$= \frac{2\pi}{3}\rho a^3.$$

5. The moment of inertia of the spherical shell is

$$\int_B \rho(x^2+y^2)\,dV,$$

choosing the z-axis as axis.



Step 2: $\displaystyle\int_B \rho(r^2\sin^2\theta)(r^2\sin\theta\,dr\,d\theta\,d\phi)$

Step 3: The θ limits are $\theta = 0$ and $\theta = \pi$.

The ϕ limits are $\phi = 0$ and $\phi = 2\pi$.

The r limits are $r = b$ and $r = a$.

Step 4: $\displaystyle\int_{r=b}^{r=a}\left(\int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi}\rho r^4\sin^3\theta\,d\theta\right)d\phi\right)dr$

$$= \int_{r=b}^{r=a}\left(\int_{\phi=0}^{\phi=2\pi}\frac{4}{3}\rho r^4\,d\phi\right)dr$$

$$= \int_{r=b}^{r=a}\frac{8\pi\rho r^4}{3}\,dr$$

$$= \frac{8\pi\rho}{15}(a^5-b^5).$$

This is the value quoted in Section 2.

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